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PARAMETRIC RESONANCE IN GUN TUBES

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This work examines the likelihood of encountering parametric resonance in gun tubes. The resonance is induced conceptually by the periodic changes in transverse stiffness induced by

(i) the axial vibrations resulting from a single application of over

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ballistic pressure - "single round parametric resonance".

(ii) the periodic applications of ballistic pressure such as encountered in an automatic weapon - "multiple round parametric resonance".

Results show that ballistic cycles currently employed in the 60mm MCAAAC semi automatic cannon are not likely to excite single round resonance. Unusually brief cycles, however, are shown to be capable of producing resonance amplifications of three orders of magnitude in less than twenty cycles of axial vibration. By proper design of the pressure cycle and/or the fundamental axial frequency of the tube, this type of resonance is rather easily avoided.

Further results show that for the 20mm M139 machine gun, amplifications in excess of fifty can be reached in under five seconds of continuous firing. A special application of the work of Krajcinovic and Herrmann leads to a set of instability contours from which the growth (characteristic) exponent can be determined as a function of the ratio of natural and excitation frequencies and the product of the ballistic impulse and the tube slenderness ratio. Control or elimination of multi-round resonance can be maintained either through control of the initial conditions or by designing for mismatch between the transverse frequencies and integral mulitples of one-half the excitation frequency, i.e., the firing rate.

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INTRODUCTION AND BACKGROUND

Forced vibrations of undamped linear systems are characterized by the differential equation:

$$\ddot{x} + \omega^2 x = f \quad ; \quad \omega^2 = k/m \tag{1}$$

where m and k are the inertia and stiffness parameters of the system and x represents the system displacement. Conventionally m and k are constants and f is a time-variant force which causes resonance if it contains a component having the system period $2\pi/\sqrt{k/m}$. The resonant term is linear in the time variable t and is a particular solution of (1).

If ω is time dependent the solution of equation (1) is much more complicated and in most cases has only been achieved through approximate methods or numerical quadrature. An important subclass of problems exists, however, for which a good deal of theoretical progress has been made. These are problems in which the variation of ω is <u>periodic</u> and f is identically zero. Such cases are represented by the <u>homogeneous</u> linear differential equation:

$$\ddot{x} + \omega^2(t)x = 0 \tag{2}$$

where $\omega^2 = \omega_0^2(1 - \varepsilon\Phi(t))$

and Φ is periodic in time. Since this equation is homogeneous it admits a general solution of the form:

$$x = Ax_1(t) + Bx_2(t) \tag{2a}$$

where A and B depend only on the initial conditions of the problem. Floquet's theorem[1] allows for two solutions of the form:

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[1]Morse, P. M. and Feshbach, H., <u>Methods of Theoretical Physics</u>, McGraw Hill, 1953, 557.

$$x_1(t) = \phi_1 e^{\gamma t}$$

$$x_2(t) = \phi_2 e^{-\gamma t} \tag{3}$$

Historically, equation (2) is known as Hill's Equation and (3) are its Floquet solutions. (See Appendix A.) The $\phi_i(t)$ have the same periodicity as the 'excitation' function Φ (t). Thus if γ has a non-zero real part, one of these solutions is unstable and the general solution exhibits exponential growth provided that the initial conditions are not those which would cause the corresponding coefficient of the growth term to vanish. Theoretically it has been shown[2] that unstable solutions can result whenever the ratio of a system natural frequency to the frequency of excitation takes on values in the neighborhoods of integral multiples of one-half (cf Fig. 1). Thus the primary instability, for example, will be encountered when the excitation frequency approaches twice a natural frequency of the system. We therefore have three fundamental differences between conventional forced-resonance and that induced parametrically:

(a) Forced resonance is independent of the initial conditions whereas parametric resonance is not. Given a force component operating at a natural frequency of a system, resonance must occur whereas a parameter (stiffness, mass) varying periodically at 2/n times the system frequency (n an integer) need not produce resonance

[2]Bolotin, V. V., The Dynamic Stability of Elastic Systems, Holden-Day, 1964, 22-23.

if the initial conditions can be controlled. This is especially significant when the short term response of a system is of interest for there is a wide choice of initial conditions for which the early response will have decreasing amplitude.

(b) Forced resonance consists of oscillations whose amplitude increases linearly with time whereas parametric resonance produces exponential growth.

(c) Forced resonance occurs if and only if the forcing frequency exactly equals a natural frequency of the system. In contrast parametric resonance can occur whenever an integral multiple of the excitation frequency approaches twice the value of a natural frequency. That is, parametric resonance - unlike forced resonance - is not a singular phenomenon but occurs throughout the neighborhood regions of a countable infinity of critical frequency ratios. It is therefore a regional phenomenon. An infinity of unstable regions exist, the most important of which is the primary region of instability.

Effect of Linear Damping

The addition of the linear damping term, $2c\dot{x}$, into equation (2) creates no complication since a transformation $x = vw$ can always be found (even when c is time dependent) - such that $w = e^{-\int c dt}$ and v solves the differential equation:

$$\ddot{v} - [c^2 + \dot{c} - \omega^2(1 - \varepsilon\Phi(t))]v = 0$$

which, if $\ddot{c} = 0$, has the form of equation (2) and hence has solutions (3). Thus

$$x = vw = A\phi_1(t)e^{[\gamma t - \int c dt]} + B\phi_2(t)e^{-[\gamma t + \int c dt]}$$

i.e., the inclusion of a linear damping term results in a simple subtraction from γ.

Two Examples of Hill's Equation

Several examples of Hill's equation are given in the literature[2,3]. The one most often cited pertains to a beam column subjected to a periodically varying axial compressive load P(t) such as depicted in Figure a. below:

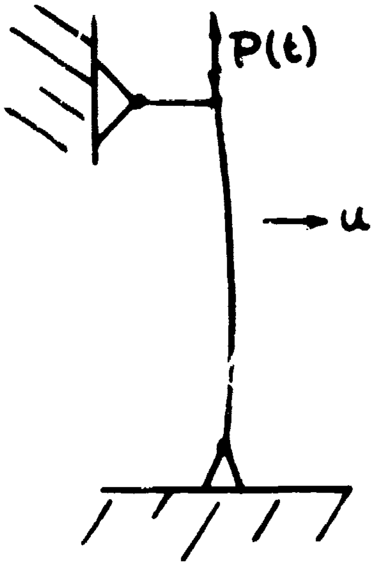

Figure a. - Classical Beam Problem Governed by Hill's Equation

The governing differential equation from Euler-Bernoulli beam theory is:

$$EI \frac{\partial^4 u}{\partial x^4} + P(t) \frac{\partial^2 u}{\partial x^2} + \rho \frac{\partial^2 u}{\partial t^2} = 0 \qquad (4)$$

As Krajcinovic and Hermann[5] have pointed out, an attempt to separate variables through the substitution $u(x,t) = X(t)f(t)$ will result in the ordinary differential equation:

---

[2] Bolotin, V. V., The Dynamic Stability of Elastic Systems, Holden-Day, 1964, 22-23.

[3] Den Hartog, J. P., Mechanical Vibrations, McGraw Hill, 1940, 378.

[5] Krajcinovic, P. P. and Hermann, G., Stability of Straight Bars Subjected to Repeated Impulsive Compression, AIAA Journal, Oct 68, 2025-2027.

$$\frac{EI}{\rho}\frac{X^{iv}}{X} + \frac{P(t)}{\rho}\frac{X''}{X} = -\frac{\ddot{f}}{f}$$

Since the right hand side of this equation is independent of x:

$$\frac{EIX^{iv}}{\rho X} = \text{const} = \alpha$$

and

$$\frac{X''}{\rho X} = \text{const} = \alpha\beta$$

Thus $$\ddot{f} + \alpha[1 - \beta P(t)]f = 0 \tag{5a}$$

and $$EIX^{iv} - \alpha\rho X = 0 \tag{5b}$$

and $$EIX^{iv} + \frac{1}{\beta}X'' = 0 \tag{5c}$$

While equation (5a) is the desired Hill's equation for the system, the separation of variables approach is only valid when X(x) also satisfies equations (5b) and (5c); i.e., the modes of free vibration which solve (5b), must be identical to the buckling modes which solve (5c). The case depicted in figure a. - a hinged-hinged support system - does in fact satisfy both of these conditions. In most cases, however, the boundary conditions lead to modes which do not satisfy both (5b) and (5c). In such cases an approximate Hill's equation can be obtained through a variational procedure, such as that due to Galerkin. In either case, therefore, the problem is reduced to the analysis of equation (5a) where α and β derive from an analysis which is either exact or approximate.

Another example - one which more directly leads to Hill's equation - considers a system with time-variant inertia, such as a child pumping a swing (figure b). Essentially a concentrated mass is raised and lowered periodically along a relatively massless rod (or chain, etc.) which pivots at 0. The rotational inertia of the pendulum thus varies periodically in time. The example is one where the path taken by the mass through the gravitational field results in a net amount of work per cycle of the motion.

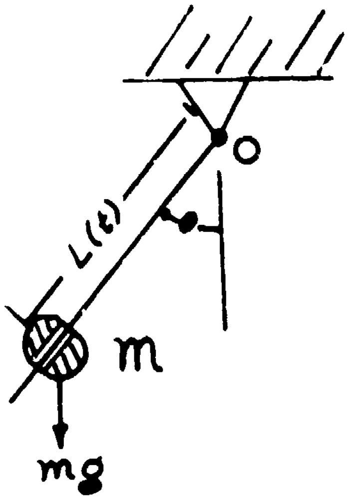

Figure 6. - Swing Problem Leading to Hill's Equation

The equation of motion:

$$-\ell(t)mg\sin\theta = \frac{d}{dt}(m\ell^2(t)\dot{\theta})$$

$$= 2m\ell\frac{d\ell}{dt}\frac{d\theta}{dt} + m\ell^2\frac{d^2\theta}{dt^2} \qquad (6)$$

Approximating: $\sin\theta \sim \theta$,

$$\ell(t)\frac{d^2\theta}{dt^2} + 2\frac{d\ell}{dt}\frac{d\theta}{dt} + g\theta = 0$$

Defining $\ell = \ell_0+h(t)$, $\tau = \omega t$, and $\omega^2 = g/\ell_0$ leads to the nondimensional equivalent of equation (6):

$$\left(1 + \frac{h(\tau)}{\ell_0}\right)\ddot{\theta} + \frac{2}{\ell_0}\dot{h}\dot{\theta} + \theta = 0 \quad ; \quad (\dot{\ }) = d/d\tau$$

If $h(\tau) = \ell_0\varepsilon\cos 2\tau$ and only first order terms in ε are retained:

$$\ddot{\theta} - 4\varepsilon\sin 2\tau\dot{\theta} + (1 - \varepsilon\cos 2\tau)\theta = 0 \qquad (7)$$

Through the afore-mentioned transformation $\theta = vw$, the equation for v and w are:

$$\ddot{v} + (1 + 3\varepsilon \cos 2\tau)v = 0 \qquad \text{(Hill's Equation)}$$

and

$$w = e^{-\varepsilon \cos 2\tau}$$

A great amount of consideration has been given[6] to cases where the 'characteristic exponent' γ, is purely imaginary and the periodic excitation Φ (t) is sinusoidal. Equation (2) then reads:

$$\ddot{x} + (a - 2q \cos 2\tau)x = 0 \qquad (8)$$

This equation - a special case of Hill's equation - is called Mathieu's Equation (canonical form). As with any Hill's equation, Mathieu's equation yields periodic solutions (called Mathieu functions) corresponding to purely imaginary, rational values of the characteristic exponent γ. With a view toward special armament applications, however, this report will deal only with the unstable solutions of Hill's equation, i.e., those cases in which γ is real.

PARAMETRIC EXCITATION - ARMAMENT

There are at least two possible sources of parametric excitation in gun tubes - that is, two ways in which periodic coefficients can be introduced into the beam equations of motion. The most obvious can be called 'multiple round excitation' and derives from the periodicity present in automatic weapons in which several time-variant forces operate at the firing rate of the weapon. A reasonably comprehensive

[6] McLachlan, N. W., Theory and Application of Mathieu Functions, Oxford Clarendon Press, 1947.

differential equation including these forces was derived in a previous report[7]. Figure 2 shows a cantilevered beam model of a gun tube acted upon by several curvature-induced loads (constant projectile velocity is assumed for simplicity). In general one observes several time dependent coefficients multiplying the various displacement derivatives. In automatic weapons, these coefficients are reproduced periodically according to the firing rate and will appear in the Hill's equation obtained upon integration of the space variable. In this report the effects of only one such term will be investigated - namely, that corresponding to the periodic ballistic pressure applied axially at the breech.

A second and less obvious cause of parametric excitation derives from the coupling between axial and transverse tube vibrations. The simplest equation incorporating the necessary nonlinear coupling terms was derived by McIvor and Bernard[8] in 1973. Essentially the idea is that a single impulsively applied load will set a column ringing with free axial vibrations. Nonlinear terms - oscillating at the frequency of these vibrations, couple with the transverse displacement variables through the stiffness coefficients. We can call this 'single round excitation'. Thus kinetic energy from the axial vibrations can feed transverse modes and lead to parametric resonance. The governing

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[7] Simkins, T., Pflegl, G., Scanlon, R., Dynamic Response of the M113 Gun Tube to Travelling Ballistic Pressure and Data Smoothing as Applied to XM150 Acceleration Data, WVT-TR-75015.

[8] McIvor, J. K., and Bernard, J. E., The Dynamic Response of Columns Under Short Duration Axial Loads, Trans ASME, September 1973, 688.

differential equations were solved by the authors through a Galerkin procedure for the special case of a simply supported beam subjected to an axial end load of short duration. [It should be noted, however, that there is no guarantee that the variational quantity employed will indeed admit an extremum when the associated differential operator is nonlinear.] Since a good deal of energy is apt to be cons...ed in rigid body recoil in armament applications, fixed supports are to be avoided. Consequently a tube (beam) cantilevered from end supports which allow axial movement (Figure 5) was chosen as the subject of analysis for this report. (Relative motion of the support is ignored.)

Evidence of Parametric Resonance in Gun Tubes

In order to minimize shot dispersion in automatic weapons the current design handbook[9] dealing with gun tube design advises that the ratio of the fundamental transverse frequency of the tube to the firing rate be kept greater than 3.5. The basis for this value is a plot of shot dispersion vs. frequency ratio R_f appearing in the handbook and reproduced as figure 3 of this report. Referring to this figure three very prominent maxima are observed at successive integral values of R_f = 1,2 and 3. The reference cited in connection with this plot is a 1955 report by Wente, Shoenberger and Quinn of Purdue University[10]. Their results, shown in figure 4, are in marked contrast to those of figure 3, however. Absent is the maximum at R_f = 3.0 shown in figure 3.

[9] AMCP 706-252, Engineering Design Handbook, Gun Series, Gun Tubes, February 1964.

[10] Wente, B. E., Schoenberger, R. L., and Quinn, B. E., An Investigation of the Effort of the National Frequency of Vibration of the Barrel Upon the Dispersion of an Automatic Weapon, Purdue U., 1955, AD64132.

It is also noted that figure 4 contains no information below the value $R_f = 1.0$. Thus the only features common to both figures are apparently the maxima at $R_f = 1.0$ and $R_f = 2.0$. Accepting these maxima as the only credible information to be gleaned from the reference publications one searches for an explanation as to their cause. While the maximum at $R_f = 1.0$ may be attributed either to parametric or to ordinary (forced) resonance, that at $R_f = 2.0$ cannot be due to ordinary resonance and may be evidence of parametric resonance - which, as previously discussed, can be expected to occur near nominal values of $R_f = 1/2, 1, 3/2, \ldots. n/2, \ldots.$ Though parametric resonance should also produce a dispersion maximum at $R_f = 1.5$ in Wente's plot, it may be that it has been missed due to the paucity of data points.

Equations of Motion

The model chosen to represent armament applications is shown in figure 5. The equations of motion which include coupling between transverse and axial displacements are those of McIvor and Bernard[8]. Eliminating their dissipation parameter for simplicity, these are:

$$\ddot{u} - [u' + \frac{1}{2} v'^2]' = 0 \tag{9a}$$

$$\ddot{v} + \alpha^2 v^{iv} - (u'v')' = 0 \tag{9b}$$

Boundary Conditions

$$P(\tau)/EA = \varepsilon_a(0,\tau) = [\frac{\partial u}{\partial s} + \frac{1}{2} (\frac{\partial v}{\partial s})^2]_{0,\tau} \qquad \text{a}$$

$$0 = \varepsilon_a(1,\tau) = [\frac{\partial u}{\partial s} + \frac{1}{2} (\frac{\partial v}{\partial s})^2]_{1,\tau} \qquad \text{b}$$

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[8]McIvor, J. K., and Bernard, J. E., The Dynamic Response of Columns Under Short Duration Axial Loads, Trans ASME, September 1973, 688.

$$0 = v(0,\tau) \qquad c$$

$$0 = v''(1,\tau) \qquad d$$

$$0 = v'(0,\tau) \qquad e$$

$$0 = v'''(0,\tau) \qquad f$$

where: $u = \eta/L$ = dimensionless axial displacement

$v = \xi/L$ = dimensionless transverse displacement

$s = x/L$ = dimensionless coordinate

$\alpha^2 = I/AL^2$ = square of the reciprocal of the slenderness ratio

$\tau = at/L$ = dimensionless time variable

$a = (EA/\rho)^{1/2}$ = extensional wave speed

E = Young's Modulus of Elasticity

A = Beam Cross sectional area

ρ = mass per unit length

I = area moment of inertia

$P(\tau)$ = end loading, a ballistic pressure function of duration τ_0

ε_a = axial strain including lowest ordered nonlinearity

Multiplying (9a) and (9b) by δu and δv respectively and integrating over the length of the beam:

$$\int_0^1 (\ddot{u}\delta u - [u' + \frac{1}{2} v'^2]'\delta u)ds = 0 \qquad (10a)$$

$$\int_0^1 (\ddot{v}\delta v - [u'v']'\delta v + \alpha^2 v^{iv}\delta v)ds = 0 \qquad (10b)$$

Using boundary conditions a and b together with an integration by parts, (10a) becomes:

$$\int_0^1 \{\ddot{u}\delta u + [u' + \frac{1}{2} v'^2]\delta u'\}ds + \frac{P(\tau)}{EA} \delta u(0,\tau) = 0 \tag{11a}$$

Similarly, boundary conditions b through f applied to (10b) yield:

$$\int_0^1 (\ddot{v}\delta v + u'v'\delta v' + \alpha^2 v''\delta v'')ds + \frac{1}{2} v'(1,\tau)^3 \delta v(1,\tau) = 0 \tag{11b}$$

The last term in (11b) is an order higher than those retained and is therefore ignored. Except for the boundary terms, equations (11) are identical to those obtained by McIvor and Bernard for the case of a simply supported beam under end loading. Our boundary conditions, however, dictate a choice of completely different approximating functions in the Galerkin approach to solution. Since both ends of the beam in our problem are free to move axially, we must choose functions which do not constrain the function u at the end points (0,1), i.e., there are no geometric constraints such as are present in the problem solved by McIvor and Bernard. A set of functions which appear to satisfy these requirements:

$$u = \sum_j q_j(\tau) \cos j\pi s \quad ; \quad j = 0,1,2,... \tag{12}$$

For the transverse motion, the eigenfunctions of a cantilevered beam satisfy the conditions c thru f. Hence:

$$v = \sum_m T_m(\tau) W_m(s) \quad ; \quad m = 1,2,3,... \tag{13}$$

where

$$W_m(s) = \cosh\beta_m s - \cos\beta_m s - \frac{\cosh\beta_m + \cos\beta_m}{\sinh\beta_m + \sin\beta_m}(\sinh\beta_m s - \sin\beta_m s)$$

and the β_m have values 1.875, 4.694, 7.855, etc.; further values can be found in any standard vibrations textbook (cf ref 11).

Substitution of (12) and (13) into equations (11) and making use of the orthogonality of the trial functions and the independence of the variational quantities δq_j and δT_m leads to the following sets of nonlinearly coupled, ordinary differential equations:

$$j = 0 : \qquad \ddot{q}_0 = -P(\tau)/EA \tag{14}$$

$$j = 1,2,\ldots \qquad \ddot{q}_j + (j\pi)^2 q_j + \sum_{m,n} A_{jmn} T_m T_n = -2P(\tau)/EA \tag{15a}$$

$$m = 1,2,\ldots \qquad \ddot{T}_m + \alpha^2\beta_m^4 T_m + \sum_{j,n} A_{jmn} q_j T_n = 0 \tag{15b}$$

where the A_{jmn} are defined from the integration:

$$\int_0^1 v'^2 \delta u' ds = -\pi\int_0^1 \sum_m T_m W_m' \sum_n T_n W_n' \sum_{j=1,2,\ldots} j\delta q_j \sin j\pi s ds \equiv \sum_{j,m,n} A_{jmn} T_m T_n \delta q_j$$

Table I gives the values of these coupling coefficients through A_{666}. Equation (14) is decoupled from the others and being representative of rigid body motion is of no further interest. Equations (15) can be solved numerically by any of several numerical integration programs[12] once α has been specified along with the nondimensional load function $P(\tau)/EA$.

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[11] Nowacki, W., Dynamics of Elastic Systems, Chapman and Hall, 1963, 122.

[12] Ralston and Wilt, Mathematical Methods for Digital Computations, Wiley, 1960, 95-109.

RESONANCE

During parametric resonance certain transverse coordinates $T_m(\tau)$ become much larger than the rest. Thus the quadratic coupling terms can be ignored except those expected to resonate. In <u>single round resonance</u> it is expected that the response to $P(\tau)$ will mainly occur in the $q_1(\tau)$ variable (i.e., the fundamental axial mode) and therefore only one such term need be considered. If m = M represents the particular transverse mode such that $\alpha\beta_M{}^2 = \pi/2$, primary parametric resonance will result. That is, the natural frequency $\alpha\beta_M{}^2$ is half the frequency of the exciting variable $q_1 = A\sin\pi\tau$, $\tau \geq \tau_0$, where τ_0 is the duration of the load pulse $P(\tau)$.

For the study of single round parametric resonance therefore, equations (15) reduce to:

$$\ddot{q}_1 + \pi^2 q_1 = -2P(\tau)/EA \equiv -2P^*(\tau) \; ; \quad \tau \leq \tau_0 \tag{16a}$$

$$\ddot{T}_M + (\alpha^2\beta_M{}^4 + A_{1MM}q_1(\tau))T_M = 0 \tag{16b}$$

In <u>multiple round resonance</u>, transient axial vibrations are ignored and the $q_j(\tau)$ are assumed to follow periodic applications of the load function <u>quasi-statically</u>. In this case, if the firing rate is twice the Mth transverse frequency, parametric resonance will result. Since <u>all</u> of the $q_j(\tau)$ are periodic according to the firing rate, it is not acceptable to retain only the term in q_1 as in the case of single round resonance. However, the amplitudes of these quasi-static responses attenuate as $1/j^2$ (neglecting the quadratic term in 15a) and only a few need be retained for accuracy. In place of equations (15) we therefore have:

$$\ddot{q}_j + (j\pi)^2 q_j = -2P(\tau)/EA \equiv -2P^*(\tau) \tag{17a}$$

$$\ddot{T}_M + (\alpha^2\beta_M{}^4 + \sum_j A_{jMM} q_j(\tau))T_M = 0 \tag{17b}$$

where $\frac{\alpha\beta_M{}^2}{2\pi} \doteq$ (weapon firing rate)/2

The periodic character of the q_j in either (16b) or (17b) qualifies these as Hill's equations. As earlier indicated, the general solution to Hills' equation can be written as:

$$T_M(\tau) = a\phi_1(\tau)e^{\gamma\tau} + b\phi_2(\tau)e^{-\gamma\tau} \tag{18}$$

where the ϕ_i are periodic functions having the period of q_j.

It should be mentioned in passing that when γ is real, a plot of the solution (18) will always show oscillations near the natural frequency $\alpha\beta_M{}^2$ even though the ϕ_i have the same period as q, i.e., approximately half the natural period $\frac{2\pi}{\alpha\beta_M{}^2}$. That is, the periodicity is not representative of the oscillatory appearance of the response.

It can be shown (see Appendix A) that once a solution for Hill's equation is obtained over one period of the excitation, it can be extended analytically for all time by means of Floquet's theorem. Further, in the case of Mathieu's equation, analytical methods have been developed leading to the direct determination of γ, the characteristic exponent. A detailed series of curves for this purpose were developed by S. J. Zaroodny[13] in 1955.

[13] Zaroodny, S. J., An Elementary Review of the Mathieu-Hill Equation of a Real Variable Based on Numerical Solutions, Ballistic Research Laboratory Memo. Report 878, Aberdeen Proving Ground, MD, 1955.

EXAMPLE I - THE MCAAAC GUN TUBE

It is of interest to assess the likelihood of encountering single round parametric resonance in the particularly long and slender tube planned for the Medium Caliber Anti-Armor Automatic Cannon* currently in the design stage. To this end equations (16) will be employed after first establishing the magnitude of $P^*(\tau)$ and the mode M in which resonance might be expected.

Equation (16b) always possess the (trivial) solution $T_M = 0$. This solution only applies when the 'initial' conditions of displacement and velocity following the application of the load $P(\tau)$ are identically zero in which case the response T_M will be null no matter how intense the 'excitation' $q_j(\tau)$. This is of little concern in armament applications since a good deal of transverse motion is certain to be excited by the firing of a round. For example the recoil of a slightly curved gun tube or the motion of the projectile therein will always excite some non-zero 'initial' motion. The axial vibration $q_1(\tau)$ in response to the ballistic pressure pulse will generally result in amplification of these initial motions by its appearance in equation (16) if the parameter [illegible] happens to be 'tuned' for parametric resonance.

According to results from the latest finite element (NASTRAN) model of the MCAAAC tube (see Table II), the fundamental axial frequency is very close to being twice the frequency of the fifth transverse mode -

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* The MCAAAC concept plans for a two or three round 'burst' and is therefore not an automatic weapon in the same sense as a conventional machine gun. Thus multi-round resonance is thought not to apply and the investigation is confined to that of resonance which might be induced from the firing of a single round.

making primary parametric resonance of this mode a possibility if enough axial excitation is produced from the application of ballistic pressure to the breech. The parameters for a uniform beam model of this tube were chosen, therefore, so that the fundamental axial frequency of 580 hz corresponds to π in equation (16a) and exactly half this value corresponds to $\alpha\beta_5^2$ of equation (16b). A summary of the pertinent parameters implied by these assumptions appears in Table III. The load function is approximated by a haversine shape. Thus (16a) reads:

$$\ddot{q}_1 + \pi^2 q_1 = -P_o^*(1 - \cos\frac{2\pi\tau}{\tau_o}) \; ; \quad \tau \leq \tau_o$$

The response following the termination of ballistic pressure is sinusoidal with amplitude:

$$q_1(\max) = \left| 2P_c^* \left(\frac{\sin\pi\tau_o/2}{1-(\frac{\tau_o}{2})^2}\right) \right|$$

Assuming negligable response from the other axial modes, equation (16) becomes (for suitable choice of 'initial' time zero):

$$\ddot{T}_5 + \alpha^2\beta_5^4(1 + \varepsilon\cos\pi\tau)T_5 = 0$$

$$\text{where } \varepsilon = A_{155}q_1(\max)/\alpha^2\beta_5^4$$

Using values from Tables I and III, ε is evaluated at 2.07×10^{-2}. The solution for T_5 is given by expression (18). In view of the smallness of ε and the precise state of tuning assumed, a very good approximation for γ can be obtained by the method of strained parameters[14]. The result is:

$$\gamma \doteq \pi\varepsilon/8$$

14Nayfeh, A. H., Perturbation Methods, John Wiley, 1973, 63.

In real time the magnitude of this exponent (see Table III) becomes:

$$\frac{\pi\varepsilon}{8} \times 1160 \doteq 9.44$$

In general the coefficient of the growth term in (18) depends on both the initial displacement and velocity of the Mth transverse mode at 'time zero'. It is this quantity which is amplified, the remaining term of (18) becoming less important as time moves on. As a specific example it is notable that for <u>certain</u> initial conditions (see Appendix A), the general solution (18) degenerates to:

$$T_M(\tau) = C\phi_1(\tau)e^{\gamma\tau} \quad ; \quad \gamma > 0$$

In this case any transverse initial displacement $T_5(0)$ is amplified according to the multiplier $e^{\gamma\tau}$. The real time computed value of 9.44 implies an amplification factor of nearly three orders of magnitude only 3/4 seconds after excitation. In practice, however, this build-up is unlikely owing to the attenuating effect of damping and the improbable state of tuning and initial conditions necessary for maximum γ.

The probability of experiencing single round parametric resonance can be significantly increased if the duration of the ballistic cycle becomes briefer than that assumed. Actually, it is the ratio of the period of the ballistic pulse - whatever its shape - to the fundamental axial period which is of importance. Figure 6[15] shows the

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[15] Harris, C. M., and Crede, C. E., <u>Shock and Vibration Handbook</u>, Vol. I - <u>Basic Theory and Measurements</u>, McGraw-Hill, 1961, 8-24.

enormous influence of this ratio on the value of γ. For example, a haversine pulse of ratio 0.8 will solicit an axial response [$q_1(max)$] which is about twenty-five times greater than the previous case (where the ratio was approximately 3.5). Since γ directly depends on $q_1(max)$, an amplification of three orders of magnitude is realized in less than seventeen cycles of axial vibration (about 30 milliseconds). In view of this potential for large γ it is therefore important that the design of a weapon be such that axial vibration magnitudes be kept small - principally by creating intentional mismatch between the fundamental axial period and the period of the ballistic cycle. Referring again to figure 6, a haversine ballistic period should be at least twice the fundamental axial period. For ballistic pulse shapes which deviate considerably from a haversine, a response spectrum similar to that of figure 6 can be easily derived via computer.

Before moving on to a specific example in multiple round resonance, some consideration should be given to the manner in which the excitation differs from that considered in the single round situation. As previously stated it is not the free axial vibrations but rather the quasi-static responses of the axial modes which serve as excitation of transverse vibrations. For example if $P^*(\tau)$ is a single haversine pulse of duration τ_0:

$$\text{i.e.} \quad P^*(\tau) = P^*_0/2(1 - \cos 2\pi\tau/\tau_0)$$

then the solution to (17a) is:

$$q_j(\tau) = \frac{-P_0^*}{(j\pi)^2}\left[1 - \frac{\tau_0^{\,2}}{\tau_0^{\,2}-(\frac{2}{j})^2}\cos\frac{2\pi\tau}{\tau_0} + \frac{(\frac{2}{j})^2}{\tau_0^{\,2}-(\frac{2}{j})^2}\cos\pi j\tau\right] \qquad (19)$$

If $\tau_0 >> 2/j$, the natural period, then

$$q_j \doteq -2P^*(\tau)/(j\pi)^2 \qquad (20)$$

that is, the response is quasi-static. In all weapons of short tube length - typically automatic small arms - the ballistic period is much longer than the fundamental natural period of axial vibration and the assumption, $\tau_0 >> 2/j$ is justified. Thus the oscillatory terms in the solution (19) can be neglected with little sacrifice to accuracy. On the other hand while τ_0 may be long compared to the fundamental axial period, it is very short when compared to the periods of the first few transverse modes. It is tempting, therefore, to consider replacing the $q_j(\tau)$ - as they appear in (17b) - by impulsive type terms. It will be shown that such a replacement sacrifices little in the way of accuracy and leads to a very convenient consolidation of results.

Multiple Round Parametric Resonance - Impulse Excitation

In 1968, D. P. Krajcinovic and G. Herrmann published work[5] dealing with the parametric resonance of bars subjected to repeated impulsive compression. The equation studied by the authors can be obtained by substituting the quasi-static solutions of (17a) into (17b):

$$\ddot{T}_M + \Omega_M^{\,2}[1 - \nu_M P^*(\tau)]T_M = 0 \qquad (21)$$

[5] Krajcinovic, P. P. and Herrmann. G., Stability of Straight Bars Subjected to Repeated Impulsive Compression, AIAA Journal, Oct. 68, 2025-2027.

where
$$\Omega_M^2 = \alpha^2\beta_M^4 \qquad \text{and} \qquad \nu_M = \sum_j 2A_{jMM}/(j\pi\Omega_M)^2$$

The load function considered by Krajcinovic and Herrmann:

$$P^*(\tau) = P_1^* + P_0^* \sum_{k=-\infty}^{k=\infty} \delta(\theta\tau - k\theta\bar{\tau}) \tag{22}$$

where $\bar{\tau}$ is the period of $P^*(\tau)$. Physically, (22) represents an infinite series of impulsively applied compressive loads superposed upon a steady load of magnitude P_1^*. $\delta(z)$ represents the Dirac function and θ is a parameter having the dimensions of frequency. In practice the authors did not have to deal with the full load expression (22) but only one cycle thereof, for example - a single impulse applied at $\tau = \bar{\tau}$. This is possible because Floquet theory (see Appendix A) enables the solution over one period of the excitation to be extended indefinitely in time. Furthermore, questions of stability can be answered by considering whether the motion grows or decays as a result of a single load application. Thus the load function actually used for analysis was equivalent to:

$$P^*(\tau) = P_0^*\delta[\theta(\tau - \bar{\tau})] \tag{23}$$

that is, a single impulsively applied load at $\tau = \bar{\tau}$.

Defining $\nu_M P_0^* = \mu_M$, equation (21) becomes:

$$\ddot{T} + \Omega^2[1 - \mu\delta(\theta(\tau - \bar{\tau}))]T = 0 \tag{24}$$

(The subscript M is implied throughout.)

Stability Analysis

Floquet's theorem[16] guarantees the existence of particular solutions $T(\tau)$, to equation (24) such that

$$T(\tau + \bar{\tau}) = \rho T(\tau) \tag{25}$$

where $\bar{\tau}$ is the period of the applied load. We are interested in determining the conditions under which ρ may be real and have an absolute value greater than unity indicating a growth of the response amplitude after one period of the excitation. Bolotin[17] has shown that if two linearly independent solutions $T_1(\tau)$ and $T_2(\tau)$ are chosen which satisfy the so-called 'unitary conditions':

$$\begin{aligned} T_1(0) &= 1 \qquad \dot{T}_1(0) = 0 \\ T_2(0) &= 0 \qquad \dot{T}_2(0) = 1 \end{aligned} \tag{26}$$

then the equation for ρ becomes simply:

$$\rho^2 - 2A\rho + 1 = 0$$

where

$$A = \frac{1}{2}[T_1(\bar{\tau}^+) + \dot{T}_2(\bar{\tau}^+)]$$

The condition that a value of ρ exists with $|\rho| > 1$, therefore amounts to:

$$|A| > 1 \tag{27}$$

It is also apparent that the two roots of the quadratic equation are reciprocal - i.e., $\rho_1\rho_2 = 1$.

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16 Meirovitch, L., Methods of Analytical Dynamics, McGraw-Hill, 1970,264.

17 Bolotin, V. V., The Dynamic Stability of Elastic Systems, Holden-Day, 1964, 14.

Now two linearly independent solutions to equation (24) which satisfy the unitary initial conditions (26) are:

$$T_1(\tau) = \cos \Omega\tau$$

$$T_2(\tau) = \frac{1}{\Omega}\sin \Omega\tau \tag{28}$$

$$\ldots\tau \leq \bar{\tau}$$

In order to use the inequality (27), the values of $T_1(\bar{\tau}^+)$ and $\dot{T}_2(\bar{\tau}^+)$ are required. While the functions themselves are continuous at $\tau = \bar{\tau}$, their derivatives are not - owing to the application of the impulse at this instant. Thus the derivative, $\dot{T}_2(\bar{\tau})$ cannot be obtained by simply substituting into the derivative expression for T_2. The <u>step change</u> in the derivative, however, can be computed by a direct integration of equation (24) over a short time interval containing the point $\tau = \bar{\tau}$.

i.e. $$\dot{T}\Big]_{\bar{\tau}-\varepsilon}^{\bar{\tau}+\varepsilon} = \int_{\bar{\tau}-\varepsilon}^{\bar{\tau}+\varepsilon} \Omega^2[\mu\delta(\theta(\tau - \bar{\tau})-1)]T(\tau)d\tau$$

Since $T(\tau)$ is continuous at $\bar{\tau}$, the term $\Omega^2 T$ contributes nothing to the integral as ε is made vanishingly small. Thus

$$\dot{T}\Big]_{\bar{\tau}-\varepsilon}^{\bar{\tau}+\varepsilon} = \frac{\Omega^2\mu T(\bar{\tau})}{|\theta|} \quad (\varepsilon \to 0) \tag{29}$$

The right hand side of (29) derives from the relation:

$$\int_{-\infty}^{\infty}\delta(az)\phi(z)dz = \frac{1}{|a|}\int_{-\infty}^{\infty}\delta(z)\phi(\frac{z}{a})dz = \frac{1}{|a|}\phi(0)$$

where $\phi(z)$ is a test function. Since θ is arbitrary it can always be chosen as positive and the absolute signs omitted accordingly.

Thus, from (29):

$$\dot{T}_2(\bar{\tau}^+) = \frac{\Omega^2\mu}{\theta} T_2(\bar{\tau}) + \dot{T}_2(\bar{\tau}^-) = \frac{\Omega^2\mu}{\theta} T_2(\bar{\tau}) + \cos \Omega\bar{\tau}$$

hence

$$|A| = \frac{1}{2}\left| 2\cos \Omega\bar{\tau} + \frac{\Omega\mu}{\theta} \sin \Omega\bar{\tau}\right|$$

Thus the instability condition (27) becomes:

$$|A| = \left|\frac{\Omega\mu}{2\theta} \sin \Omega\bar{\tau} + \cos \Omega\bar{\tau} \right| > 1 \qquad (30)$$

In the work performed by Krajcinovic and Herrmann, Θ is defined as being inversely proportional to $\bar{\tau}$ ($\theta = \frac{2\pi}{\bar{\tau}}$) which is the customary definition when treating continuous excitation functions. Adhering to this definition in the case of repetitive, discontinuous loads, however, leads to a load function which is physically improbable. For the load function (23):

i.e. $P^*(\tau) = P_0^*\delta[\theta(\tau - \bar{\tau})]$

the impulse of this load under the above definition of θ becomes:

$$P_0^* \int_{-\infty}^{\infty} \delta[\theta(\tau - \bar{\tau})]d\tau = \frac{P_0^*}{\theta} = \frac{P_0^*\bar{\tau}}{2\pi} \qquad (31)$$

that is, the impulse strength is seen to depend on $\bar{\tau}$, the period of time between impulses. This is not the physical situation of interest in armament - and many other - lications which imply a sequence of equally strong impulses <u>independent</u> of their spacing in time. Fortunately, these cases can be handled merely by redefining θ, i.e., let $\theta = \frac{1}{\tau_0}$, where τ_0 represents the duration of the ballistic load function as shown in figure c.

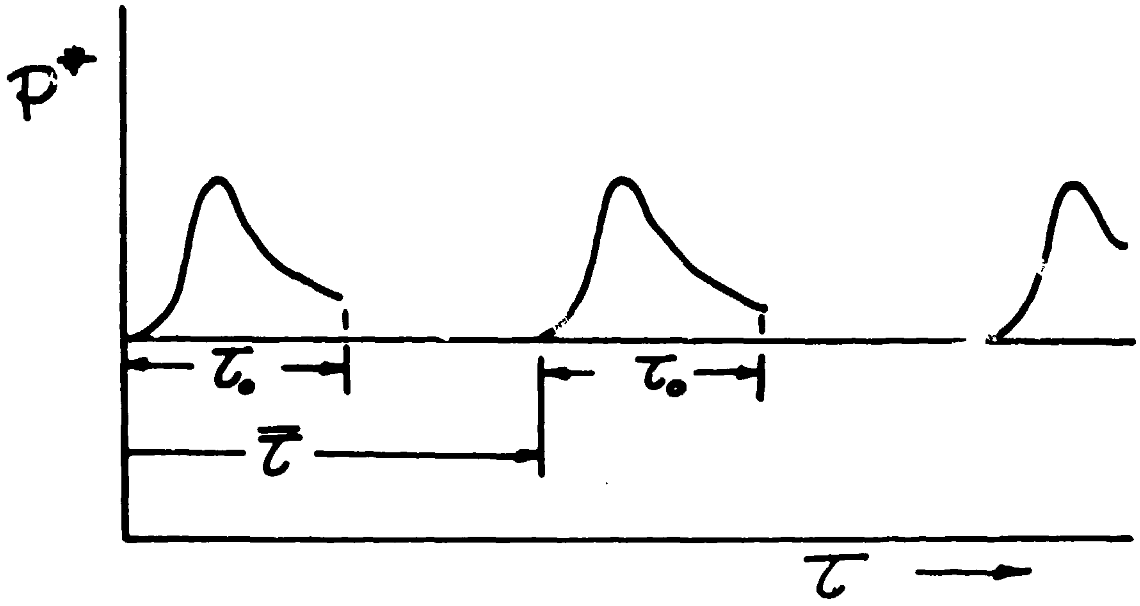

Figure c. - Periodic Ballistic Pressure Function

Under this definition of θ, the impulse (31) becomes:

$$\frac{P_0^*}{\theta} = P_0^*\tau_0 \equiv I_0$$

I_0 is defined as an impulse conveyed by a load of average value P_0^* and duration τ_0 . In practice $P^*(\tau)$ in equation (21) is to be replaced by the load function $P_0^*\delta[\theta(\tau-\bar{\tau})]$ which has the same impulse as $P^*(\tau)$. Thus P_0^* must represent the nondimensional time average axial load on the breech due to ballistic pressure.

The result of redefining θ on the instability condition (30) is considerable. Whereas the expression for A in (30) is purely a function of μ and the ratio of the natural frequency to that of the excitation, this is not the case when the meaning of θ is changed. Substituting $\theta = 1/\tau_0$ into (30) the condition for instability becomes:

$$|A| = \left|\frac{\Omega\mu\tau_0}{2} \sin \Omega\bar{\tau} + \cos \Omega\bar{\tau}\right| > 1 \tag{32}$$

Thus for the Mth equation (17b) we have $\mu = \mu_M$ and $\Omega = \Omega_M$ as previously defined so that (32) becomes finally,

$$|A_M| = \left|\sum_j \frac{A_{jMM}}{(j\pi\beta_M)^2} \frac{I_o}{\alpha} \sin \Omega_M\bar{\tau} + \cos \Omega_M\bar{\tau}\right| \equiv$$

$$\equiv \left| \frac{C_M I_o}{\alpha} \sin 2\pi \frac{\Omega_M}{\omega_e} + \cos 2\pi \frac{\Omega_M}{\omega_e} \right| \tag{33}$$

where definitions for C_M and ω_e are inferred. ω_e now represents the frequency of the impulse excitation instead of θ. A_M is seen to depend on I_o/α - the product of the impulse and the slenderness ratio - and the ratio of the unperturbed natural frequency to that of the excitation. It is also apparent that in contrast to (30), A_M is now periodic in the frequency ratio implying equal stability criteria for all 'zones' of instability $\frac{\Omega_M}{\omega_e} = 1/2, 1, 3/2, \ldots, n/2, \ldots$

<u>The Characteristic Exponent - Maximum Value</u>

The general solution to Hill's equation (17b) is given by (18). This solution, having property (25), implies the following definition for γ, the characteristic, or 'growth', exponent:

$$\gamma = \frac{\ln|\rho|}{\bar{\tau}} \quad ; \quad \gamma \text{ real}$$

For a given excitation frequency ω_e, $\bar{\tau}$ is a specified quantity and maximum values of $|\rho|$ determine maximum γ. In calculating the particular frequency ratio for which $|\rho|$ will be maximum, we first let

$$\Omega_M/\omega_e = n/2 + \delta$$

n designates the zone of instability while δ a local coordinate to be varied in searching for a maximum value of $|\rho|$. Substituting in (33):

$$A_M = [\frac{C_M I_o}{\alpha} \sin 2\pi\delta + \cos 2\pi\delta](-1)^n$$

$$\frac{\partial A_M}{\partial \delta} = 0 = (-1)^n\ [C_M I_o/\alpha \cos 2\pi\delta - \sin 2\pi\delta]$$

so that $\frac{1}{2\pi} \tan^{-1} \frac{C_M I_o}{\alpha} = \delta_1$, a value of δ for which A_M is extreme.

From (26b):

$$\rho = A_M \pm \sqrt{A_M^{\,2} - 1}$$

For extreme values of ρ:

$$\frac{\partial \rho}{\partial \delta} = \frac{\partial A_M}{\partial \delta} (1 \pm A_M(A_M^{\,2} - 1)^{-1/2}) = 0$$

Thus when $\delta = \delta_1$, ρ will also be extreme. Substituting:

$$|\rho|_{ext} = \left| \frac{1}{\cos 2\pi\delta_1} \pm \sqrt{\frac{1}{\cos^2 2\pi\delta_1} - 1} \right|$$

hence

$$|\rho|_{max} = \left| \frac{1 \pm \sin 2\pi\delta_1}{\cos 2\pi\delta_1} \right| \tag{34}$$

where the sign is chosen to agree with that of δ_1. Note that $|\rho|_{max}$ is confirmed in (34) to be completely independent of the zone number n. For practical ranges of $\frac{C_M I_o}{\alpha}$, δ_1 is small and can be considered a linear function of this parameter.

i.e. $\quad 2\pi\delta_1 = C_M I_o/\alpha$

and (34) gives: $\quad |\rho|_{max} = 1 + \left|\frac{C_M I_o}{\alpha}\right|$

or $- \ell n|\rho|_{max} = C_M I_o/\alpha$, to a high degree of accuracy. $\quad$ (35)

Figure 7 is a contour plot applicable to all instability zones. The contours are curves $\ln|\rho|$ = constant. The plot shows that the instability becomes broader in band and that for larger values of the parameter $R = C_M I_0/\alpha$ the values of $\ln|\rho|$ become less sensitive to variations in δ, i.e., the contour slopes $\frac{dR}{d\delta}$ become flatter.

It still remains to determine the conditions under which a repetitive load function can be justifiably replaced in Hill's equation by a repetitive series of impulses. It is expected that these conditions will not strongly depend on the particular shape of the function involved but rather its duration, τ_0 and the frequency, Ω, of the system on which it acts. For the sake of argument, therefore, a sequence of rectangular pulses $\Phi(\tau)$ is chosen as excitation to Hill's equation:

$$\ddot{f} + \Omega^2 (1 - \mu\Phi(\tau))f = 0$$

where

$$\Phi(\tau) = 1, \quad 0 < \tau \leq \tau_0$$
$$= 0, \quad \tau_0 < \tau \leq \bar{\tau}$$

and

$$\Phi(\tau + \bar{\tau}) = \Phi(\tau)$$

The corresponding expression for A in this case turns out to be[18]:

$$A = \left(\sin \Omega\tau_0 \cos p_1\tau_0 - \frac{(p_1{}^2+\Omega^2)}{2\Omega p_1} \cos \Omega\tau_0 \sin p_1\tau_0\right)\sin \Omega\bar{\tau} + \tag{36}$$
$$+ \left(\cos p_1\tau_0 \cos \Omega\tau_0 + \frac{(p_1{}^2+\Omega^2)}{2\Omega p_1} \sin p_1\tau_0 \sin \Omega\tau_0\right)\cos \Omega\bar{\tau}$$

where $p_1 = \Omega\sqrt{1-\mu}$

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[18] Bolotin, V. V., The Dynamic Stability of Elastic Systems, Holden-Day, 1964, 18.

It is easily verified that for small values of the quantity $\Omega\tau_o$, the expression for A above reduces to that in (32). Expanding the transcendental terms containing the parameter $\Omega\tau_o$, and retaining up to third order terms in this parameter, (36) becomes:

$$A = \Omega\mu\tau_o/2\ (1+\Omega^2\mu\tau_o/6)\ \sin\Omega\bar{\tau} + \cos\Omega\bar{\tau}$$

$$= \frac{C_M I_o}{\alpha}\left(1 + \frac{C_M I_o}{\alpha}\frac{\Omega\tau_o}{3}\right) \sin \Omega\bar{\tau} + \cos \Omega\bar{\tau}$$

It is therefore concluded that if

$$\Omega\tau_o << \frac{3\alpha}{C_M I_o} \qquad (37)$$

the approximation (33) will be good.

<u>EXAMPLE II - M139 AUTOMATIC WEAPON</u>

The 20mm M139 at first glance (Figure 8) would appear to be vulnerable to transverse vibration excitation. It has the appearance of a long and slender tube - as compared with, say, a typical .30 caliber gun such as illustrated in reference (9). Actually the slenderness ratio of the two are nearly equal having values of approximately 103 and 92 respectively. In spite of this our intuition is not in error. As previously demonstrated, it is not the slenderness ratio alone, but its product with the non-dimensional ballistic impulse which determines the strength of the exponential growth in parametric resonance. As it turns out, this impulse (I_o) for the 20mm round is nearly three times larger than that computed for the .30 cal weapon.

A complete set of calculations follow - leading to the evaluation of the parameters C_M and I_o/α. The possible values of the growth exponent γ are then determinable from figure 7 .

A uniform tube approximation is assumed:

D_o = 1.5 in - tube outer diameter

D_i = 0.79 in - tube inner diameter

L = 72 in - overall tube length

P_{AV} = 40,000 psi - time-average ballistic pressure

t_o = .0015 sec - effective duration time of ballistic pressure function

a = 2.02×10^5 in/sec - extensional wave speed

E = 30×10^6 psi - Young's Modulus of Elasticity

If the ratio of the unperturbed natural frequency Ω_M to the excitation frequency ω_e is nearly an integral multiple of 1/2, parametric resonance is possible.

$$\text{i.e.,} \quad \frac{\Omega_M}{\omega_e} \equiv \frac{\alpha\beta_M^2}{2\pi/\bar{\tau}} \doteq n/2 \qquad n = 1,2,3\ldots$$

$$\text{or } \bar{\tau} = \frac{n\pi}{\alpha\beta_M^2}$$

$\bar{\tau}$ is the time between rounds. In real time, $\bar{t} = \bar{\tau}L/a$, corresponding to a firing rate of 915 rounds per minute - a reasonable value in practice. From the previously assigned values:

$$\frac{1}{\alpha} = L\sqrt{A/I} = 103.3 \quad \text{- the slenderness ratio}$$

If we choose M = 1, that is, parametric resonance of the fundamental transverse mode of vibration, then for n = 2,

$$\bar{\tau} = \frac{2\pi}{\alpha\beta_1^2} \doteq 184 \quad \text{and} \quad \bar{t} \doteq .066 \text{ sec}$$

where we have used the value $\beta_1 = 1.875$ - corresponding to the fundamental mode of a cantilevered beam.

The nondimensional axial load (average)*:

$$P_o^* = \frac{\pi P_{AV} D_i^2}{4EA} = 5.12\times10^{-4}$$

The duration of this load in nondimensional time:

$$\tau_o = \frac{a}{L} t_o = 4.21$$

Thus the nondimensional impulse is:

$$I_o = P_o^* \tau_o = 21.58\times10^{-4}$$

Recalling the definition of C_M:

$$C_M = \sum_j \frac{A_{jMM}}{(j\pi\beta_M)^2}$$

For M = 1, therefore:

$$C_1 = \sum_j \frac{A_{j11}}{[j\pi(1.875)]^2} \doteq .234$$

The parameter $\Omega_1\tau_o$ must satisfy the smallness criterion (38); i.e.,

$$\Omega_1\tau_o << \frac{3\alpha}{C_1 I_o}$$

Substituting values for the parameters:

$$.0956 << 57.5$$

which would appear to be satisfactory.

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*Note that the real quantity of interest is the nondimensional impulse, I_o. If ballistic curves are available this quantity can be determined directly by a simple integration.

Expression (36), or equivalently figure 7 shows that the greatest value for $\ln|\rho|_{max}$ is .052. The corresponding growth coefficient:

$$\gamma_{max} = \frac{\ln|\rho|_{max}}{\bar{t}} = \frac{.052}{.066} = .786$$

This value of γ can produce a vibration amplification of over 50 in 5 seconds of repeated firing. In other words if at any time during firing the deflection of, say, the muzzle is a given amount then 5 sec later this deflection can be over 50 times greater.

It is not likely that the exact state of tuning will exist such that the maximum amplification will be realized. Furthermore, as previously shown, damping will reduce the amplification. On the other-hand there are other loads which have not been considered and which may be non-mitigating. Reactions from the moving projectile and the travelling propellant pressure as well as the relative movements of supports are synchronous with the load considered in this report and may add to its effect - possibly more than offsetting the reduction caused by damping. Again it is mentioned in connection with figure 7 that the state of tuning leading to large amplifications need not be as precise when larger values of the abcissa are encountered.

DISCUSSION

The models assumed in this report are of course, somewhat idealized. It is always a question as to when more detail should be included. How much is gained, for example, by including the geometry of a variable cross sectional area when knowledge of the support conditions is always so imprecise? Further, it is almost certain that the general character of the results shown in figure 7 will prevail regardless of the detail incorporated in the modeling effort. That is, no matter how detailed the model, parametric resonance will probably be confined to a narrow band of frequencies for the tensile loads likely to be induced on the tube through firing. The frequency is going to have to be 'just right' for it to occur. But given the proper frequency ratio our preceding analysis definitely suggests that ballistic forces are sufficient to create significant exponential growth. It is doubtful if any higher degree of confidence in these general findings could be obtained through the use of more refined models. On the other hand, the frequencies and even the particular mode which may resonate cannot be predicted with confidence unless an extremely detailed model is employed. Even then, in cases where parametric resonance is suspect, field measurements should be made to determine precisely the frequencies of free vibration and excitation. As mentioned earlier in this report, there exists certain inconclusive evidence[9,10] that dispersion maxima observed in tests with certain automatic weapons may be caused by parametric resonance. On the basis of the feasibility demonstrated herein, it is advisable that such experiments be repeated in a more tightly controlled manner so that a conclusion may finally be drawn.

9AMCP 706-252, Engineering Design Handbook, Gun Series, Gun Tubes, February 1964.

10Wente, D. E., Schoenberger, R. L., and Quinn, B. E., An Investigation of the Effort of the National Frequency of Vibration of the Barrel Upon the Dispersions of an Automatic Weapon, Purdue U., 1955, AD 64132.

TABLE I*- COUPLING COEFFICIENTS A_{jmn}

A_{111} = -9.83	A_{121} = 11.1
A_{112} = 11.1	A_{122} = -50.1
A_{113} = 7.71	A_{123} = 16.4
A_{114} = 1.91	A_{124} = 30.3
A_{115} = 1.68	A_{125} = 6.93
A_{116} = .819	A_{126} = 7.54
A_{131} = 7.71	A_{141} = 1.91
A_{132} = 16.4	A_{142} = 30.3
A_{133} = -128.	A_{143} = 18.4
A_{134} = 18.4	A_{144} = -247.
A_{135} = 69.6	A_{145} = 18.9
A_{136} = 11.3	A_{146} = 122
A_{151} = 1.68	A_{161} = .819
A_{152} = 6.93	A_{162} = 7.54
A_{153} = 69.6	A_{163} = 11.3
A_{154} = 19.0	A_{164} = 122.
A_{155} = -404.	A_{165} = 19.3
A_{156} = 19.3	A_{166} = -601.

*Certain symmetries are evident throughout the Table. No attempt has been made to make use of symmetry properties in order to shorten the tabulation.

TABLE I - COUPLING COEFFICIENTS A (cont.)

A_{211} = 9.78	A_{221} = -48.6
A_{212} = -48.6	A_{222} = 84.5
A_{213} = 26.3	A_{223} = -115.
A_{214} = 25.4	A_{224} = 45.7
A_{215} = 3.35	A_{225} = 92.8
A_{216} = 5.94	A_{226} = 18.4
A_{231} = 26.3	A_{241} = 25.4
A_{232} = -115.	A_{242} = 45.7
A_{233} = 94.5	A_{243} = -247.
A_{234} = -247.	A_{244} = 89.5
A_{235} = 61.6	A_{245} = -433.
A_{236} = 201.	A_{246} = 68.9
A_{251} = 3.35	A_{261} = 5.94
A_{252} = 92.8	A_{262} = 18.4
A_{253} = 61.6	A_{263} = 201.
A_{254} = -433.	A_{264} = 68.8
A_{255} = 85.9	A_{265} = -670.
A_{256} = -670.	A_{266} = 83.4

TABLE I - COUPLING COEFFICIENTS A (cont.)

$A_{311} = -6.53$	$A_{321} = 33.8$
$A_{312} = 33.8$	$A_{322} = -174.$
$A_{313} = -112.$	$A_{323} = 164.$
$A_{314} = 45.5$	$A_{324} = -216.$
$A_{315} = 53.4$	$A_{325} = 73.1$
$A_{316} = 2.50$	$A_{326} = 183.$
$A_{331} = -112.$	$A_{341} = 45.5$
$A_{332} = 164.$	$A_{342} = -216.$
$A_{333} = -156.$	$A_{343} = 196.$
$A_{334} = 196.$	$A_{344} = -279.$
$A_{335} = -434.$	$A_{345} = 197.$
$A_{336} = 111.$	$A_{346} = -722.$
$A_{351} = 53.4$	$A_{361} = 2.50$
$A_{352} = 73.1$	$A_{362} = 183.$
$A_{353} = -434.$	$A_{363} = 111.$
$A_{354} = 197.$	$A_{364} = -722.$
$A_{355} = -440.$	$A_{365} = 194.$
$A_{356} = 194.$	$A_{366} = -639.$

TABLE I - COUPLING COEFFICIENTS A (cont.)

A_{411} = 8.19	A_{421} = -23.1
A_{412} = -23.1	A_{422} = 108.
A_{413} = 62.5	A_{423} = -351.
A_{414} = -201.	A_{424} = 266.
A_{415} = 68.8	A_{425} = -349.
A_{416} = 91.8	A_{426} = 98.3
A_{431} = 62.5	A_{441} = -201.
A_{432} = -351.	A_{442} = 266.
A_{433} = 273.	A_{443} = -237.
A_{434} = -237.	A_{444} = 366.
A_{435} = 303.	A_{445} = -394.
A_{436} = -674.	A_{446} = 316.
A_{451} = 68.8	A_{461} = 91.8
A_{452} = -349.	A_{462} = 98.3
A_{453} = 303.	A_{463} = -674.
A_{454} = -394.	A_{464} = 316.
A_{455} = 381.	A_{465} = -588.
A_{456} = -588.	A_{466} = 375.

TABLE I - COUPLING COEFFICIENTS A (cont.)

$A_{511} = -7.19$	$A_{521} = 28.9$
$A_{512} = 28.9$	$A_{522} = -80.1$
$A_{513} = -38.5$	$A_{523} = 183.$
$A_{514} = 98.5$	$A_{524} = -586.$
$A_{515} = -316.$	$A_{525} = 390.$
$A_{516} = 96.6$	$A_{526} = -512.$
$A_{531} = -38.5$	$A_{541} = 98.5$
$A_{532} = 183.$	$A_{542} = -586.$
$A_{533} = -66.9$	$A_{543} = 401.$
$A_{534} = 401.$	$A_{544} = -324.$
$A_{535} = -339.$	$A_{545} = 535.$
$A_{536} = 420.$	$A_{546} = -542.$
$A_{551} = -316.$	$A_{561} = 96.6$
$A_{552} = 390.$	$A_{562} = -512.$
$A_{553} = -339.$	$A_{563} = 420.$
$A_{554} = 535.$	$A_{564} = -542.$
$A_{555} = -494.$	$A_{565} = 575.$
$A_{556} = 575.$	$A_{566} = -698.$

TABLE I - COUPLING COEFFICIENTS A (cont.)

A_{611} = 7.85	A_{621} = -24.7
A_{612} = -24.7	A_{622} = 101.
A_{613} = 49.3	A_{623} = -130.
A_{614} = -56.3	A_{624} = 270.
A_{615} = 142.	A_{625} = -878.
A_{616} = -456.	A_{626} = 539.
A_{631} = 49.3	A_{641} = -56.3
A_{632} = -130.	A_{642} = 271.
A_{633} = 296.	A_{643} = -1078.
A_{634} = -1078.	A_{644} = 556.
A_{635} = 552.	A_{645} = -432.
A_{636} = -459.	A_{646} = 716.
A_{651} = 142.	A_{661} = -456.
A_{652} = -878.	A_{662} = 539.
A_{653} = 552.	A_{663} = -459.
A_{654} = -432.	A_{664} = 716.
A_{655} = 750.	A_{665} = -632.
A_{656} = -632.	A_{666} = 836.

TABLE II - EIGENVALUE ANALYSIS (NASTRAN) - MCAAAC, 60MM

(Trussed Configuration)

Mode Type		Frequency (hz)
Recoil mode		1.81
Fundamental	Transverse mode	79.5
2nd	Transverse mode	100.5
3rd	Transverse mode	125.6
4th	Transverse mode	218.7
5th	Transverse mode	296.2
6th	Transverse mode	412.7
7th	Transverse mode	525.9
Fundamental	Axial mode	580.1
2nd	Axial mode	1189.2
3rd	Axial mode	1898.4
4th	Axial mode	2496.2

TABLE III - PARAMETRIC VALUES, MCAAAC TUBE

The parametric values listed below are based on the material constants E, ρ for steel, a peak ballistic pressure of 75,500 psi, and the NASTRAN-predicted frequencies for the fundamental axial vibration and the fifth transverse vibration modes.

$$\alpha^2 = 6.44\times10^{-5}$$

$$\tau/t = 1160$$

$$\frac{P(\tau)}{EA} = \frac{P_o^*}{2}\left(1 - \cos\frac{2\pi\tau}{\tau_o}\right)$$

$$P_o^* = 7.03\times10^{-4}$$

$$\tau_o = 6.96$$

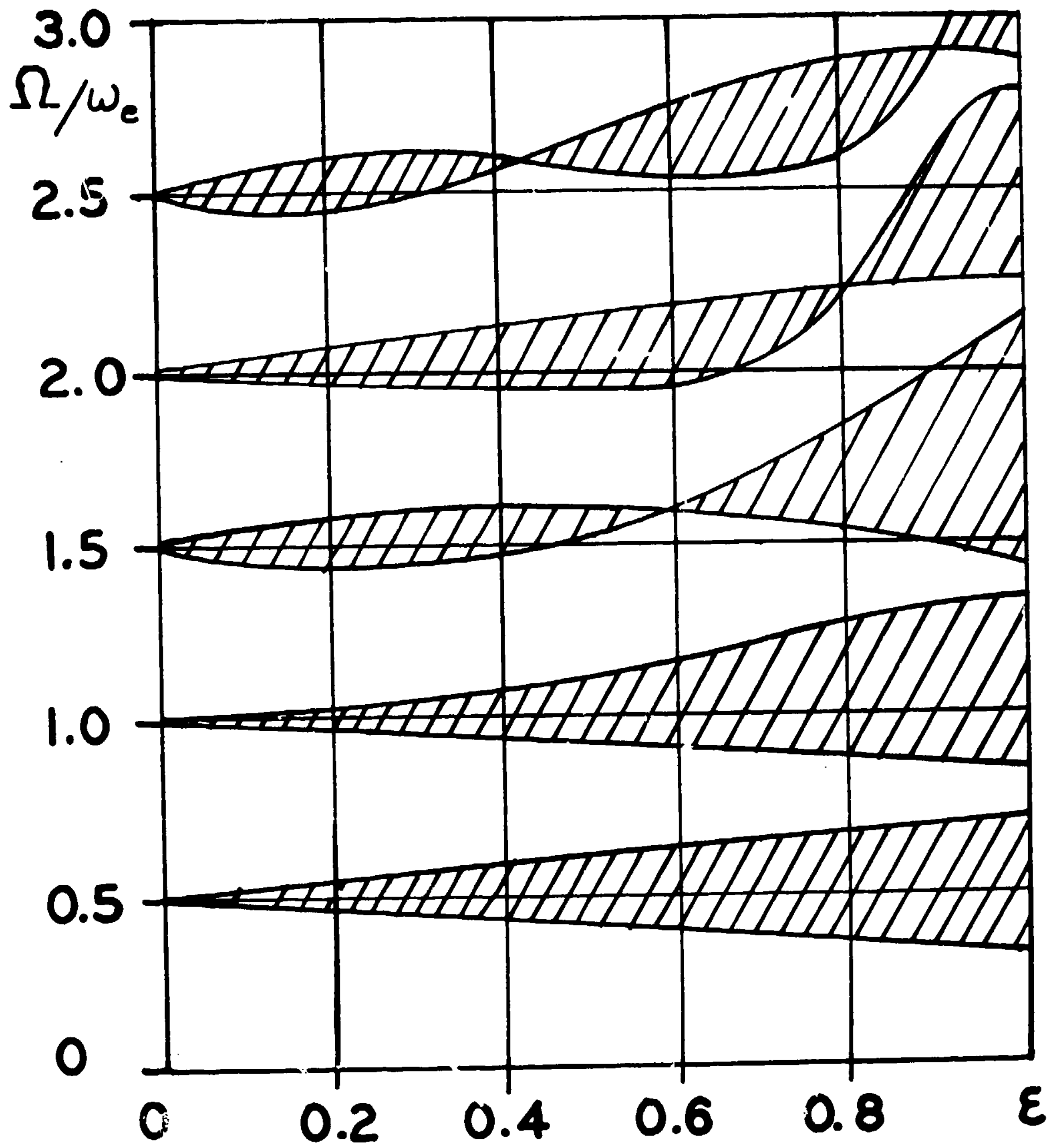
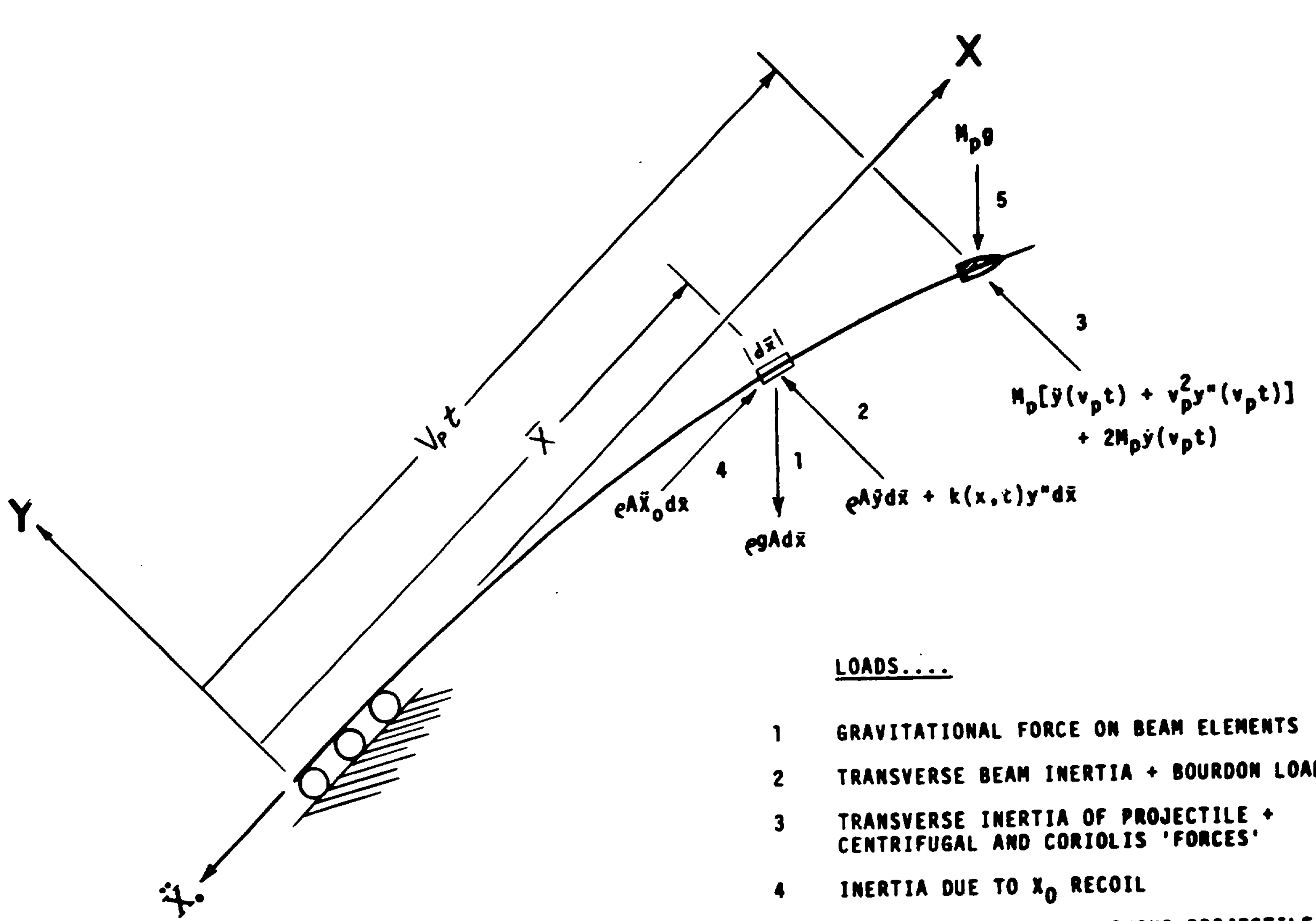


Figure 1 . .stability Zones. Excitation in Hill's Equation is a Rectangular Ripple Function. (cf ref (2), p. 18, fig. 4).



LOADS....

1 GRAVITATIONAL FORCE ON BEAM ELEMENTS

2 TRANSVERSE BEAM INERTIA + BOURDON LOADING

3 TRANSVERSE INERTIA OF PROJECTILE + CENTRIFUGAL AND CORIOLIS 'FORCES'

4 INERTIA DUE TO X_0 RECOIL

5 FORCE OF GRAVITY ON MOVING PROJECTILE

Figure 2 - Curvature Induced Loads - From Reference (7).

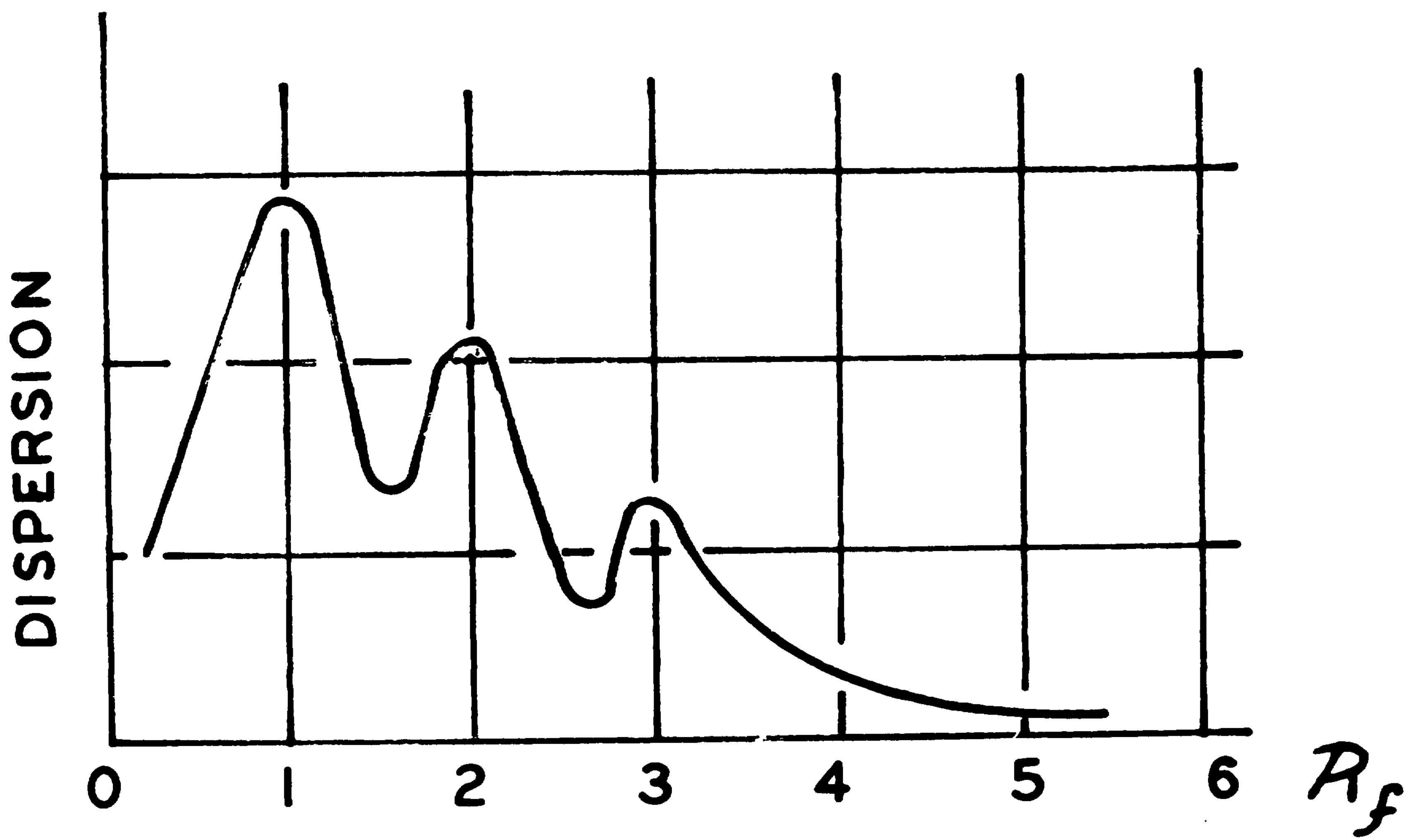


Figure 3 - Dispersion of Shot vs. Frequency Ratio (R_f = nat. frequency/firing rate) as reported in reference (9).

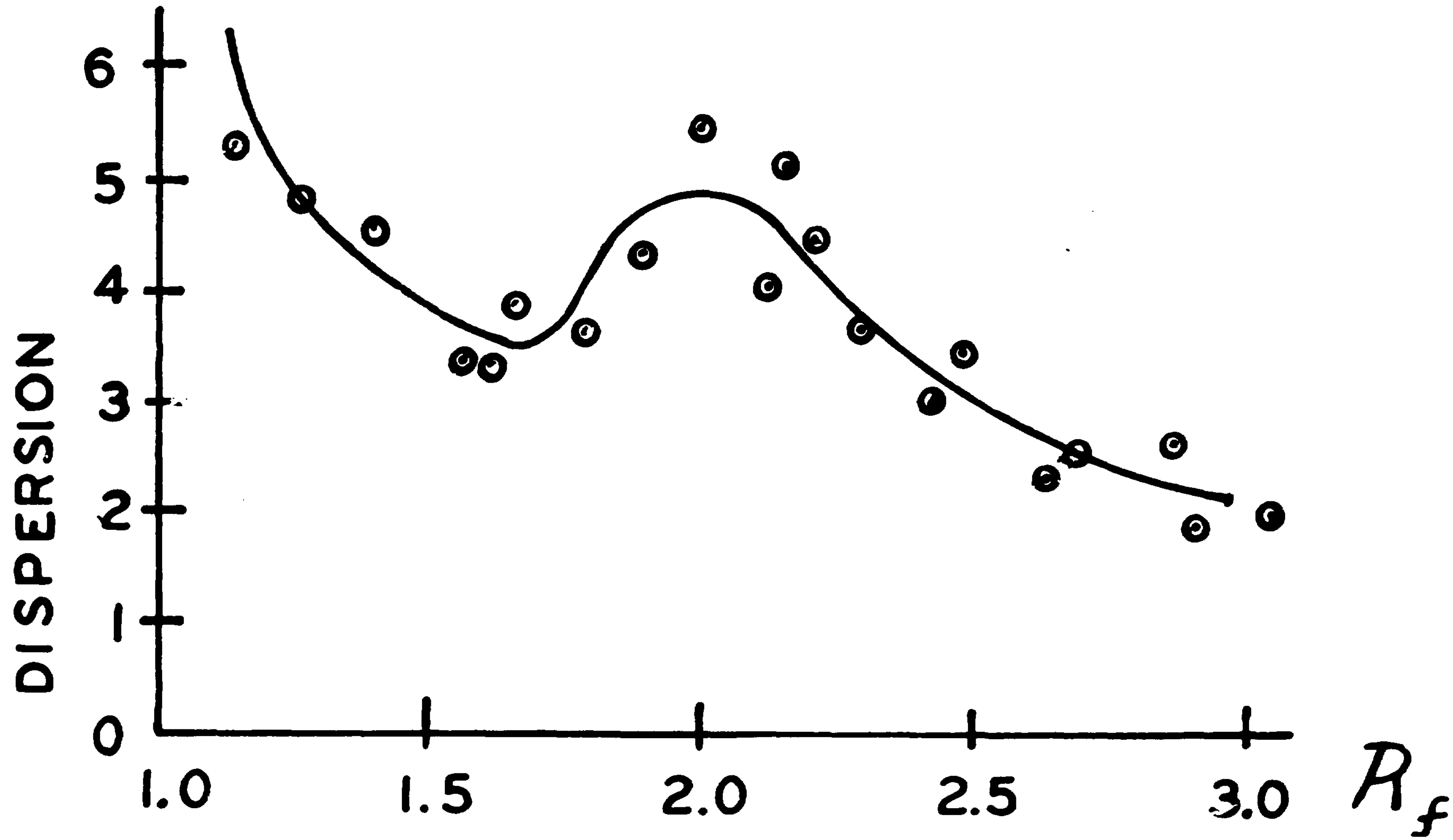


Figure 4 - Dispersion of Shot vs. Frequency Ratio - as reported in reference (10).

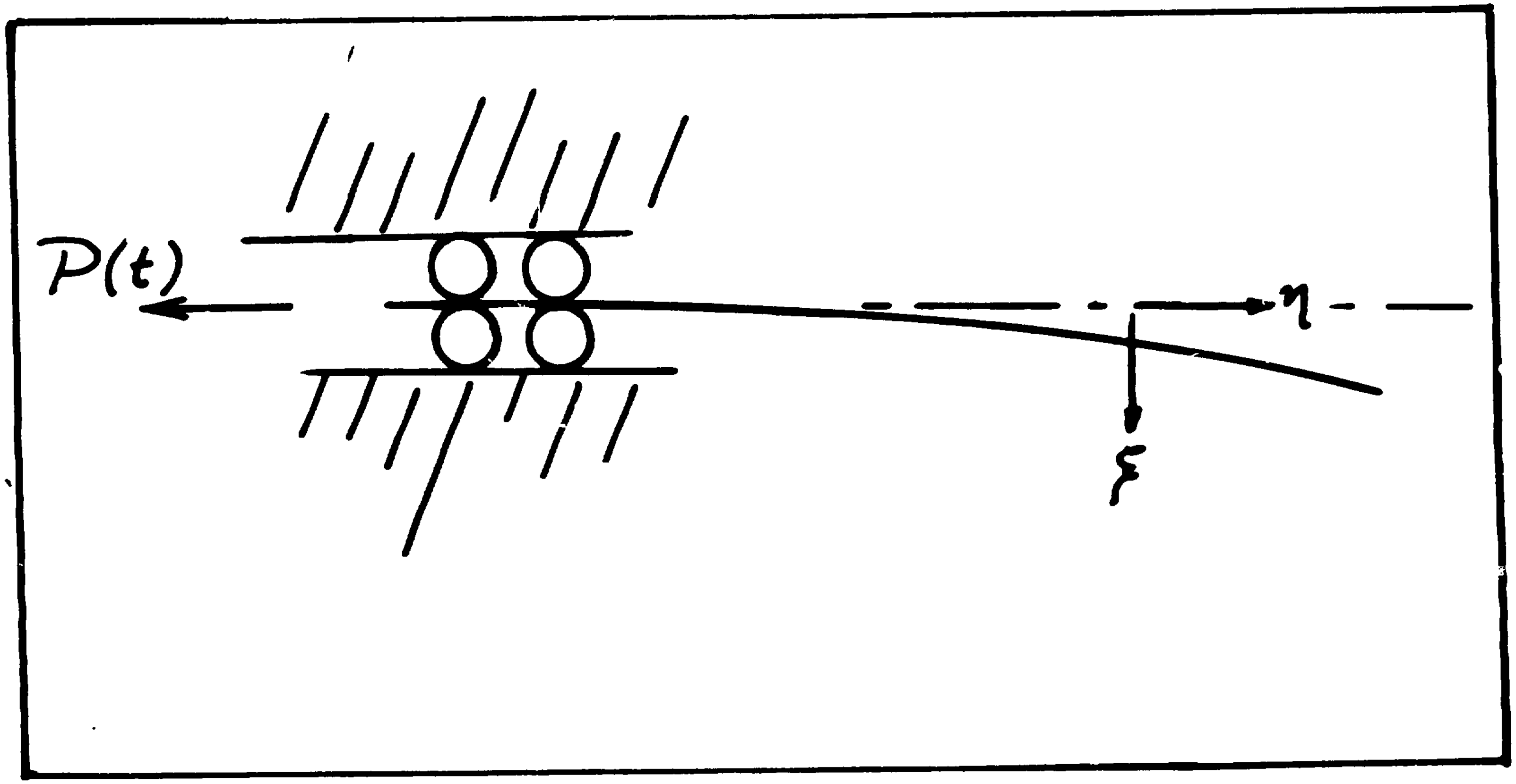


Figure 5 - Gun Tube Model - An Axially Free Cantilevered Beam

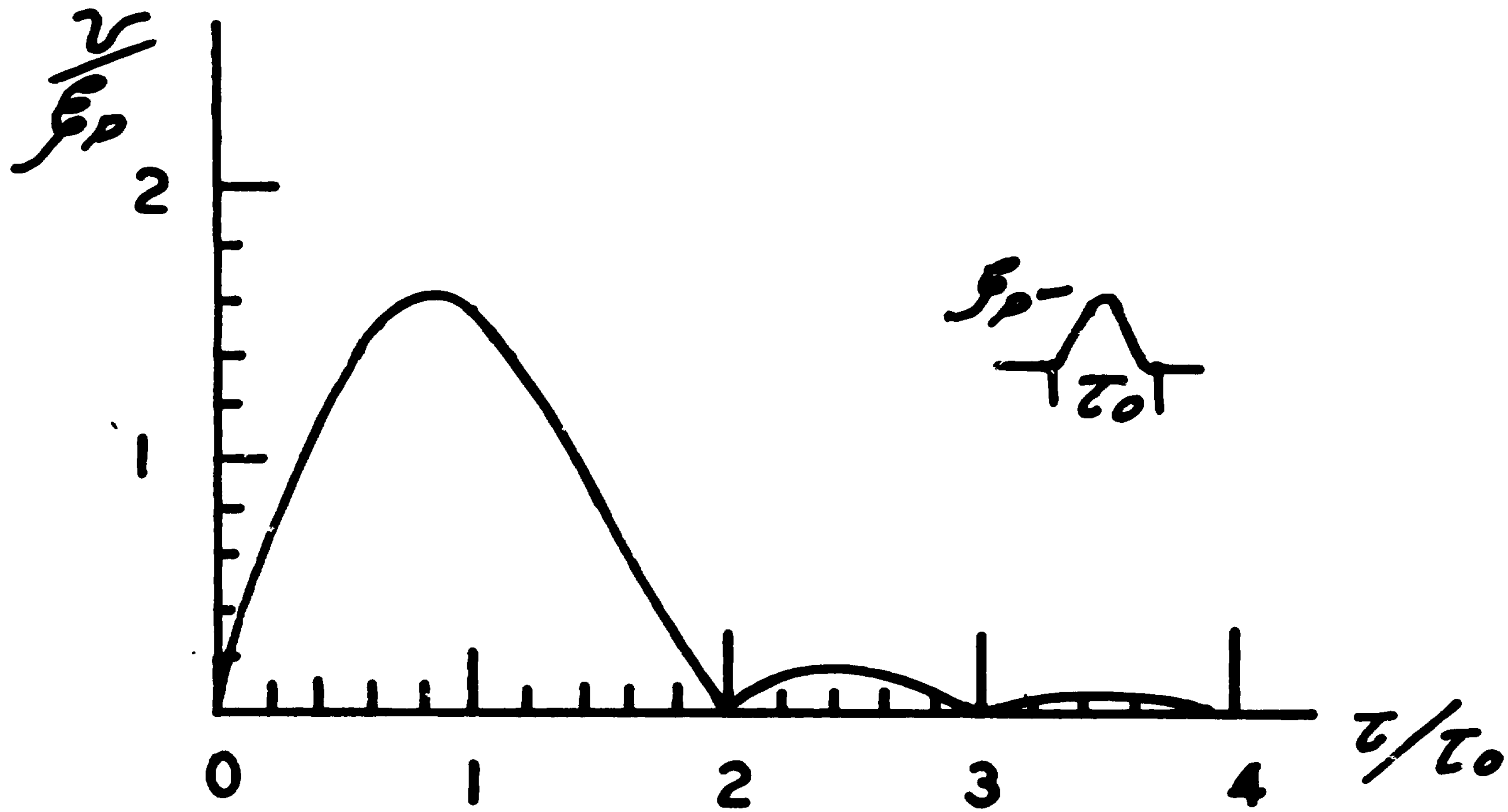


Figure 6 - Residual Response Spectra, Haversine Excitation; from Reference (15).

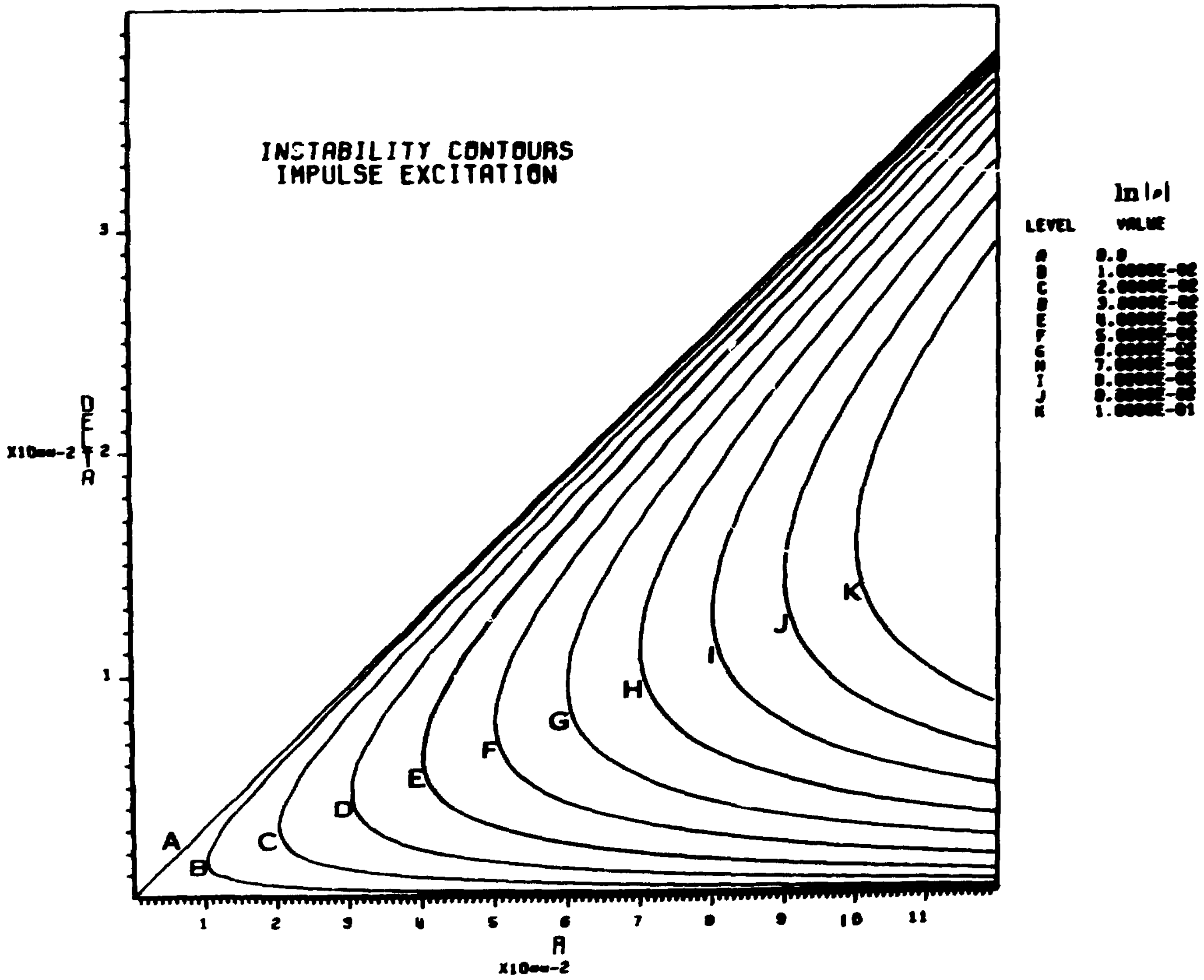
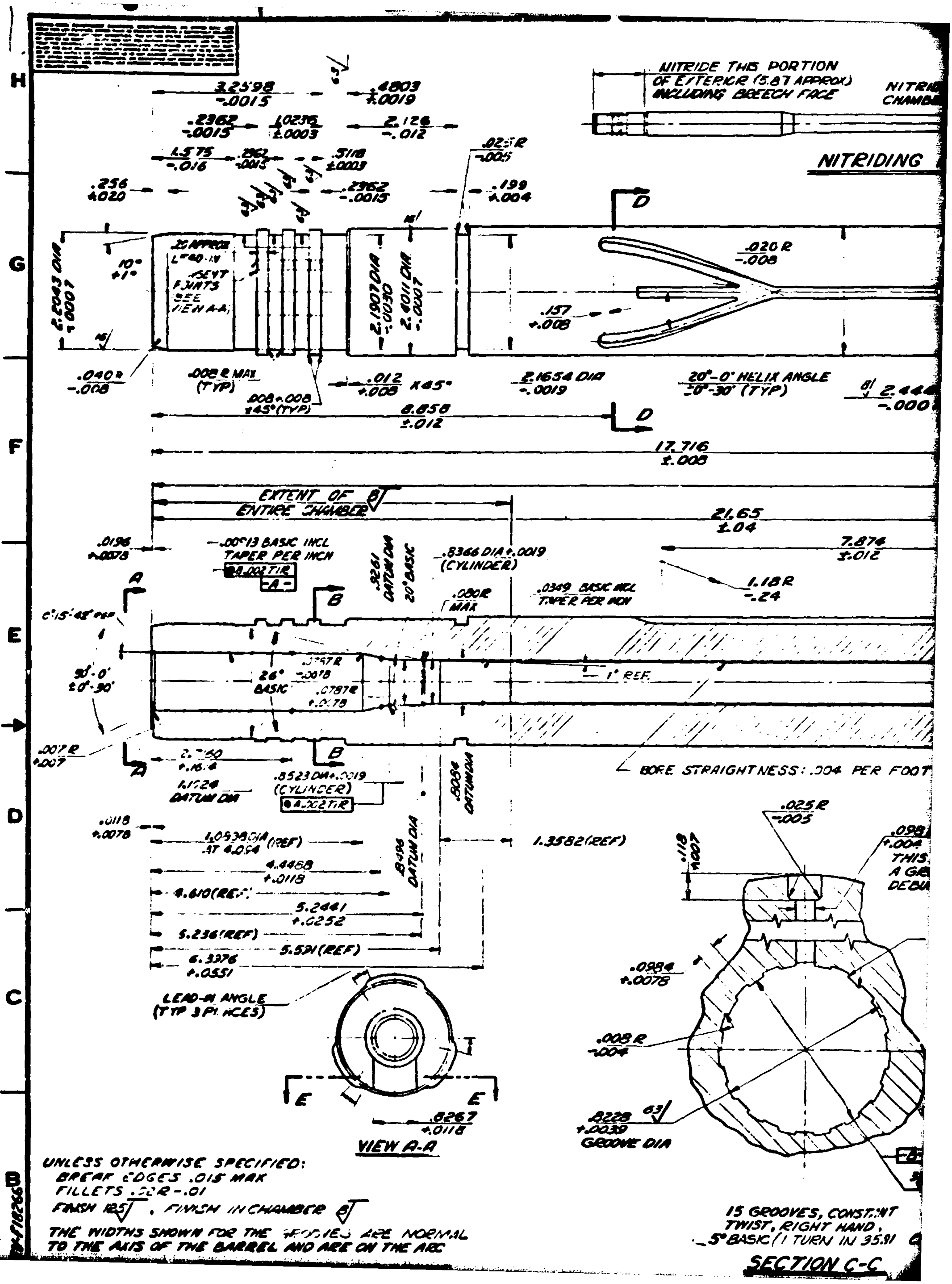
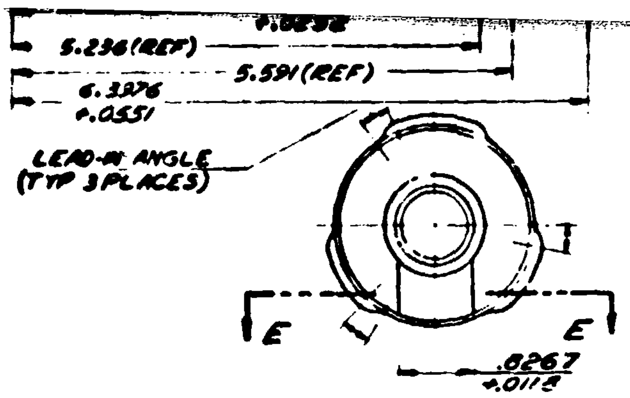


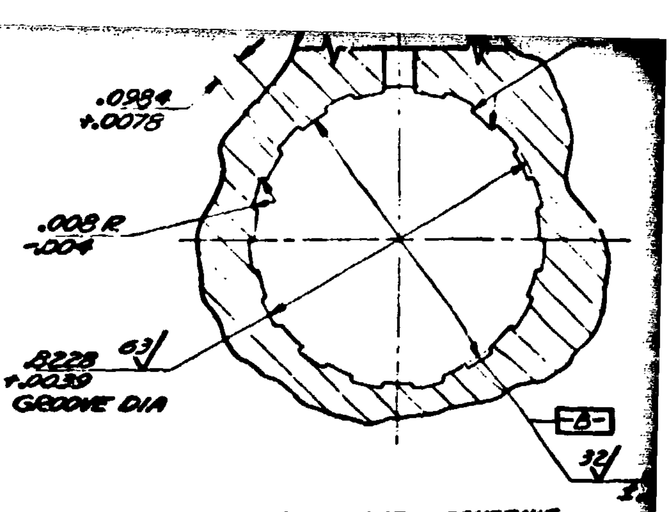
Figure 7 - Instability Contours - $R = C_M I_0/\alpha$;
$\Omega/\omega_e = n/2 + \delta$ (see text).


NITRIDE THIS PORTION OF EXTERIOR (5.87 APPROX) INCLUDING BREECH FACE
NITRIDING
3.2598 -.0015
.4803 +.0019
.2362 -.0015
.0236 ±.0003
2.126 -.012
1.575 -.016
.2362 -.0015
.5118 ±.0003
.025R -.005
.256 +.020
.2362 -.0015
.199 +.004
D
.020 R -.008
10° ±1°
2.2043 DIA +.0007
2.1907 DIA -.0030
2.4011 DIA -.0007
.157 +.008
.040R -.008
.008 R MAX (TYP)
.008+.008 x45° (TYP)
.012 +.008 X45°
2.1654 DIA -.0019
20°-0' HELIX ANGLE ±0°-30' (TYP)
8.858 ±.012
17.716 ±.008
EXTENT OF 8 ENTIRE CHAMBER
21.65 ±.04
.0196 +.0078
-.00013 BASIC INCL TAPER PER INCH
.9261 DATUM DIA
20° BASIC
.8366 DIA +.0019 (CYLINDER)
7.874 ±.012
1.18R -.24
A
B
.080R MAX
.0349 BASIC INCL TAPER PER INCH
90°-0' ±0°-30'
26° BASIC
.0787R -.0078
.0787R +.0078
1° REF.
.007 R +.007
1.1024 DATUM DIA
.8523 DIA +.0019 (CYLINDER)
.8084 DATUM DIA
BORE STRAIGHTNESS: .004 PER FOOT
.0118 +.0078
1.0538 DIA AT 4.094 (REF)
1.3582 (REF)
4.4468 +.0118
.8496 DATUM DIA
4.610 (REF)
5.2441 +.0252
5.236 (REF)
5.531 (REF)
6.3376 +.0551
.025R -.005
.118 +.007
.0984 +.0078
.008 R -.004
.8228 +.0039 GROOVE DIA
LEAD-IN ANGLE (TYP 3 PLACES)
E
.8267 +.0118
VIEW A-A
UNLESS OTHERWISE SPECIFIED:
BREAK EDGES .015 MAX
FILLETS
FINISH 125, FINISH IN CHAMBER 8
THE WIDTHS SHOWN FOR THE GROOVES ARE NORMAL TO THE AXIS OF THE BARREL AND ARE ON THE ARC
15 GROOVES, CONSTANT TWIST, RIGHT HAND, 5° BASIC (1 TURN IN 35.91
SECTION C-C
H
G
F
E
D
C
B

C



VIEW A-A



B

UNLESS OTHERWISE SPECIFIED:
BREAK EDGES .015 MAX
FILLETS .02R -.01
FINISH 125, FINISH IN CHAMBER 8

THE WIDTHS SHOWN FOR THE GROOVES ARE NORMAL TO THE AXIS OF THE BARREL AND ARE ON THE ARC

.0015 ECCENTRICITY BETWEEN BORE AND GROOVE DIAMETER OF RIFLING IS ALLOWED

FOR BARREL FORGING SEE D11576809

THE 2.2043 AND 2.9[illegible] PILOT DIAMETERS MUST BE WITHIN THE TOLERANCE SHOWN ABOVE AFTER NITRIDING. REWORK OF THESE AREAS IS PERMITTED IF NECESSARY.

A

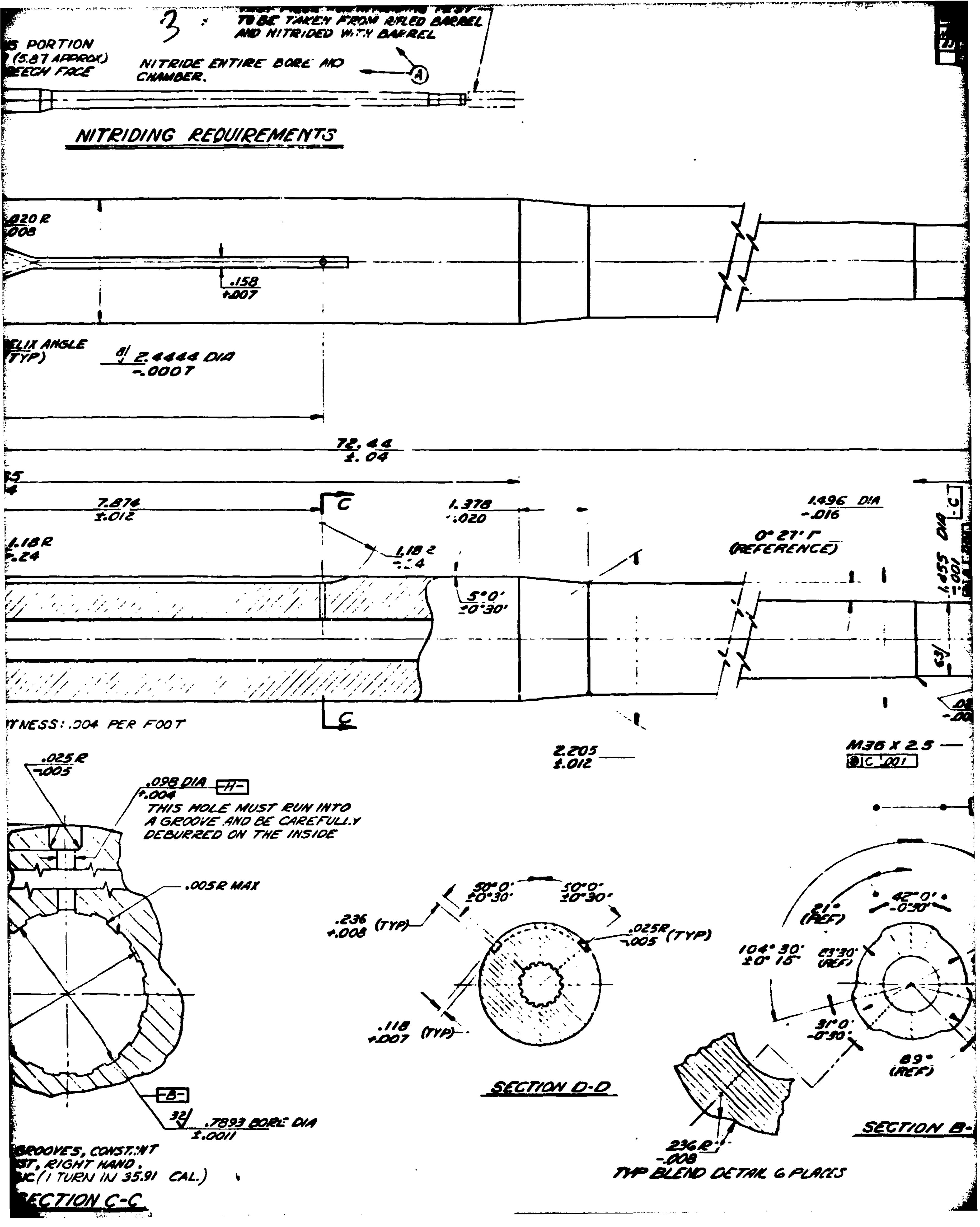
BARRELS SHALL BE NITRIDED PER DWG C11578440. SURFACES TO BE NITRIDED ARE SHOWN IN VIEW ABOVE.
MAGNETIC PARTICLE INSPECTION REQUIRED PER DWG B8768747

15 GROOVES, CONSTANT TWIST, RIGHT HAND, 5° BASIC (1 TURN IN 35.91 CAL.)

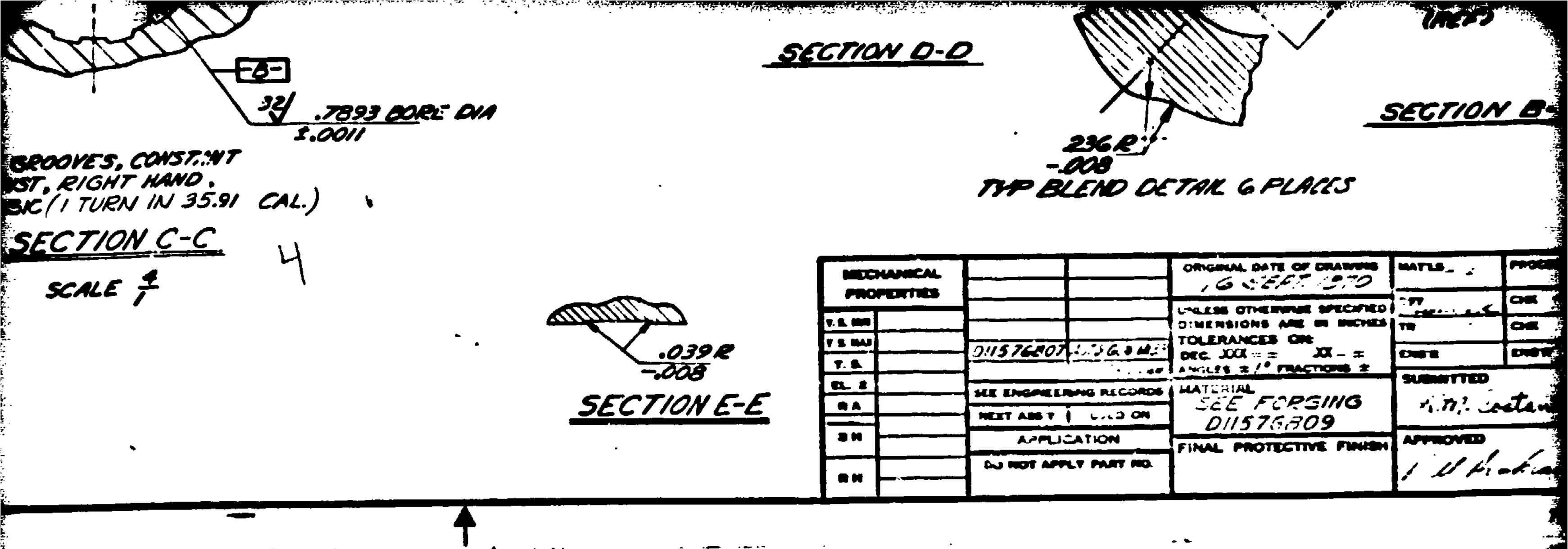
SECTION C-C

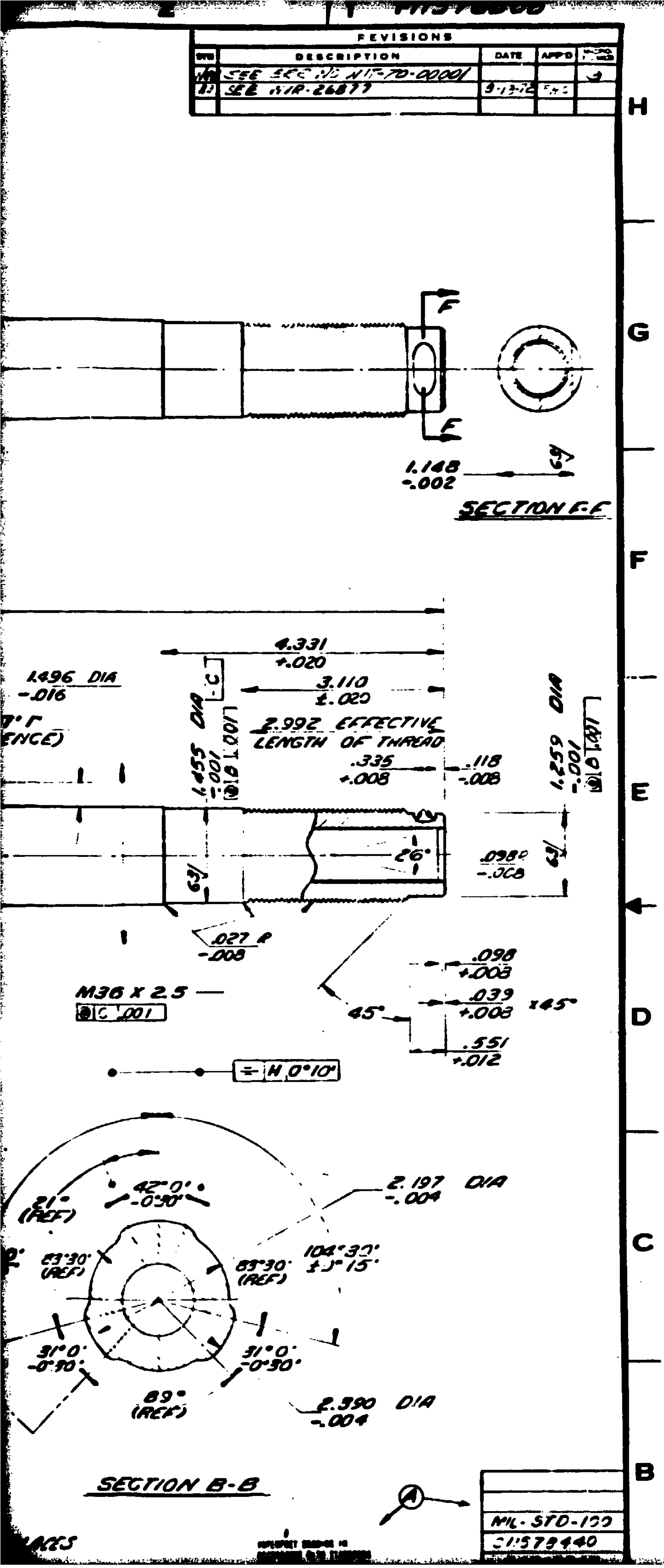
SCALE 4/1

WAS WTV-F11266

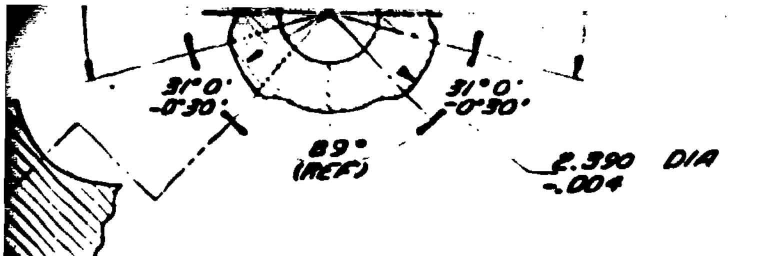


3
TO BE TAKEN FROM RIFLED BARREL
AND NITRIDED WITH BARREL
S PORTION
(5.87 APPROX)
EECH FACE
NITRIDE ENTIRE BORE AND CHAMBER.
A
NITRIDING REQUIREMENTS
.020 R
.008
.158
±.007
ELIX ANGLE
(TYP)
2.4444 DIA
-.0007
72.44
±.04
7.874
±.012
C
1.378
±.020
1.496 DIA
-.016
0° 27' F
(REFERENCE)
1.18 R
±.24
5° 0'
±0° 30'
1.455 DIA
±.001
-C-
63
TNESS: .004 PER FOOT
C
2.205
±.012
M36 X 2.5
C .001
.025 R
-.005
.098 DIA
+.004
-H-
THIS HOLE MUST RUN INTO
A GROOVE AND BE CAREFULLY
DEBURRED ON THE INSIDE
.005 R MAX
-B-
32
.7893 BORE DIA
±.0011
GROOVES, CONSTANT
ST, RIGHT HAND.
NC (1 TURN IN 35.91 CAL.)
ECTION C-C
50° 0'
±0° 30'
50° 0'
±0° 30'
.236
+.008
(TYP)
.025 R
-.005
(TYP)
.118
+.007
(TYP)
SECTION D-D
21°
(REF)
42° 0'
-0° 30'
104° 30'
±0° 15'
23° 30'
(REF)
31° 0'
-0° 30'
89°
(REF)
.236 R
-.008
TYP BLEND DETAIL 6 PLACES
SECTION B-


SECTION D-D
-B-
32
.7893 BORE DIA
±.0011
GROOVES, CONSTANT
IST. RIGHT HAND.
BIC (1 TURN IN 35.91 CAL.)
SECTION C-C
SCALE 4/1
4
.236 R
-.008
TYP BLEND DETAIL 6 PLACES
SECTION B-
.039 R
-.008
SECTION E-E
MECHANICAL PROPERTIES
Y. S. MIN
Y. S. MAX
T. S.
EL. %
R A
B H
R H
D11576807
SEE ENGINEERING RECORDS
NEXT ASS'Y
USED ON
APPLICATION
DO NOT APPLY PART NO.
ORIGINAL DATE OF DRAWING
16 SEPT 1970
UNLESS OTHERWISE SPECIFIED DIMENSIONS ARE IN INCHES
TOLERANCES ON:
DEC. XXX ± XX ±
ANGLES ± 1° FRACTIONS ±
MATERIAL
SEE FORGING
D11576809
FINAL PROTECTIVE FINISH
MAT'LS
DR
TR
CHK
CHK
ENGR
SUBMITTED
APPROVED



REVISIONS
DESCRIPTION
DATE
APP'D
SEE ECO NO NIT-70-00001
SEE NIR-26877
H
G
F
E
D
C
B
SECTION F-F
1.148
-.002
4.331
+.020
3.110
±.020
1.496 DIA
-.016
2.992 EFFECTIVE LENGTH OF THREAD
1.455 DIA
-.001
.335
+.008
.118
-.008
1.259 DIA
-.001
26°
.098
-.008
.027 R
-.008
M36 X 2.5
.098
+.008
.039
+.008
x45°
45°
.551
+.012
H 0°10'
2.197 DIA
-.004
42°0'
-0°30'
21°
(REF)
83°30'
(REF)
104°30'
±0°15'
31°0'
-0°30'
89°
(REF)
2.390 DIA
-.004
SECTION B-B
A
MIL-STD-100
21579440



SECTION B-B

TAIL 6 PLACES

MIL-STD-100
11578440
B8765747
SPEC SUMMARY (REF)

ORIGINAL DATE OF DRAWING 16 SEPT 1970	MAT'LS	PROCESS	DEPT OF THE ARMY WATERVLIET ARSENAL WATERVLIET, N.Y.
UNLESS OTHERWISE SPECIFIED DIMENSIONS ARE IN INCHES TOLERANCES ON DEC. .XXX ± ANGLES ±	DFT	CHK	BARREL, 20 MM (85 CALIBER)
	TR	CHK	
	LNGR	ENGR	
MATERIAL SEE FORGING D115 6809	SUBMITTED		
FINAL PROTECTIVE FINISH	APPROVED	F CODE 19206	11576308
		SCALE	UNIT WT SHEET OF

6

Substituting into A2:

$$Af_1(t+T) + Bf_2(t+T) = \rho[Af_1(t) + Bf_2(t)]$$

Using A3:

$$\sum_{j=1}^{2} \{Aa_{1j}f_j(t) + Ba_{2j}f_j(t)\} = \rho[Af_1(t) + Bf_2(t)]$$

Since f_1 and f_2 are independent:

$$Aa_{11} + Ba_{21} = A\rho$$

$$Aa_{12} + Ba_{22} = B\rho \qquad \text{(A5)}$$

A5 only has nontrivial solutions if the determinant of the coefficients of A and B vanish:

i.e.,
$$\begin{vmatrix} a_{11} - \rho & a_{21} \\ a_{12} & a_{22} - \rho \end{vmatrix} = 0 \qquad \text{(A6)}$$

Substituting into either of the equations A5, each root ρ_j gives a ratio B/A such that the solution will satisfy the condition A4.

$$\left(\frac{B}{A}\right)_j = \frac{\rho_j - a_{11}}{a_{21}} \qquad \text{(A7)}$$

In particular, if $f_1(t)$ and $f_2(t)$ are chosen to satisfy the so called 'unitary initial conditions':

$$f_1(0) = 1 \qquad f_2(0) = 0$$

$$f_1'(0) = 0 \qquad f_2'(0) = 1$$

then, from A2:

$$A = Y(0)$$

$$B = Y'(0)$$

and hence the ratio B/A is the ratio of initial conditions such that A4 is satisfied. Further, from A3, definitions are given to the a_{ij}:

$$a_{11} = f_1(T)$$

hence

$$a_{21} = f_2(T)$$

$$a_{12} = f_1'(T)$$

$$a_{22} = f_2'(T)$$

and $$[Y'(0)/Y(0)]_j = \frac{\rho_j - f_1(T)}{f_2(T)} \tag{A8}$$

The quadratic equation for ρ (A6) becomes:

$$\rho^2 - 2A\rho + B = 0$$

where**

$$A = \frac{1}{2}[f_1(T) + f_2'(T)]$$

$$B = f_1(T)f_2'(T) - f_2(T)f_1'(T) \tag{A9}$$

Multiplying the first of these equations by f_2 and the second by f_1 and subtracting:

$$f_1f_2'' = f_2f_1'' = 0^*$$ [19]

Integrating:

$$B = f_1(t)f_2'(t) - f_2(t)f_1'(t) = \text{const}$$

The constant has unit value in view of the unitary initial conditions employed.

19Boyce, W. E. and DiPrima, R. C., Elementary Differential Equations and Boundary Value Problems, John Wiley, 1966, 89.

*Note that this equation can be written $W_{12}' = 0$, where W_{12} is the Wronskian of $f_1(t)$ and $f_2(t)$.

**If any of the f_i have jump discontinuities at $t = T$, then $A = A(\tau^+)$ is implied.

Thus equation A6 becomes:

$$\rho^2 - 2A\rho + 1 = 0 \tag{A10}$$

whereupon

$$\rho = A \pm \sqrt{A^2 - 1}$$

Note that the two roots of this equation are reciprocal, i.e. $\rho_1\rho_2 = 1$

Thus corresponding to real roots, A4 represents a solution which grows or diminishes (according to the choice of ρ) following each period of the excitation, T. The ratio of initial conditions which will lead to either of these solutions alone is given by A8.

EXAMPLE - Delta Function Excitation

Let

$$\Phi(t) = \sum_{k=1}^{\infty} \delta(t-kT)$$

From A10, the roots ρ_j depend entirely on the quantity

$$A = \frac{1}{2}[f_1(T) + f_2'(T)]$$

where $f_1(t)$ and $f_2(t)$ are linearly independent solutions of Hill's Equation (A1) satisfying unitary initial conditions. These can be chosen as:

$$\begin{aligned} f_1(t) &= \cos \Omega t \\ f_2(t) &= \frac{1}{\Omega} \sin \Omega t \end{aligned} \tag{A11}$$

Thus $f_1(T) = \cos \Omega T$

The expression obtained by a formal differentiation of $f_2(t)$ above, is valid up to but not including time $t = T$ since the application of the first 'impulse' will cause $f_2'(T)$ to be discontinuous. This jump discontinuity can be evaluated by an integration of Hill's Equation (A1):

$$f_2'(T) - f_2'(T^-) = \lim_{\Delta t \to 0} \int_{T-\Delta t}^{T+\Delta t} \Omega^2[\varepsilon\delta(t-T)-1]f_2(t)dt = \varepsilon\Omega^2 f_2(T)$$

Thus

$$f_2'(T) = \varepsilon\Omega\sin\Omega T + \cos\Omega T$$

and

$$A = \frac{\varepsilon\Omega}{2}\sin\Omega T + \cos\Omega T$$

whereupon the roots ρ_1, ρ_2 may be evaluated.

For any given finite value of ε, intervals of the parameter ΩT exist such that these roots will be real and since their product must be unity, the following possibilities arise:

(i) $\rho_1 = \rho_2 = \pm 1$

(ii) $\rho_1 > 1$, $\rho_2 < 1$

Case (i) represent borderline situations, that is, the solutions A4 neither grow or decay with each period T of the excitation. Case (ii), on the other hand, produces one solution of the form A4 which grows in amplitude by a factor $\rho_1 > 1$ and another which decays by the factor $\rho_2 < 1$, each period of the excitation. The initial conditions which produce either of these solutions must only satisfy A8,

$$\text{i.e.} \quad \frac{Y'(0)}{Y(0)} = \frac{\rho_i - f_1(T)}{f_2(T)} = [\rho_i - \cos\Omega T]\Omega/\sin\Omega T$$

Figure A1(a,b) shows the solution when this ratio is enforced. In general, however, the solution for arbitrary initial conditions is a linear combination of these special solutions of pure growth and decay. Figure A1(c) shows one such solution exhibiting early decay eventually to be overpowered by the growing solution.

The solutions of pure growth or pure decay can be represented as:[17]

$$f_j(t) = \psi_j(t)e^{\frac{t}{T}\ln\rho_j} \tag{A12}$$

where

$$\psi_j(t+T) = \psi_j(t)$$

so that indeed,

$$f_j(t+T) = \psi_j(t)e^{(\frac{t}{T}+1)\ln\rho_j} = \rho_j f_j(t)$$

as required. In general ρ is complex

hence

$$\ln\rho_j = \ln|\rho_j| + i\arg(\rho_j)$$

Thus

$$f_j(t) = \phi_j(t)e^{\frac{t}{T}\ln|\rho_j|}$$

where

$$\phi_j(t) = \psi_j(t)e^{\frac{it}{T}\arg\rho_j} \tag{A13}$$

When the ρ_j are real, $\phi_j(t) = \psi_j(t)$ and are periodic in T. Thus the general solution to Hill's equation may be written

$$Y(t) = C_1\phi_1(t)e^{\frac{t}{T}\ln|\rho_1|} + C_2\phi_2(t)e^{\frac{t}{T}\ln|\rho_2|}$$

or, enforcing the relation $\rho_1\rho_2 = 1$:

$$Y(t) = C_1\phi_1(t)e^{\frac{t}{T}\ln|\rho_1|} + C_2\phi_2(t)e^{\frac{-t}{T}\ln|\rho_1|}$$

[17] Bolotin, V. V., The Dynamic Stability of Elastic Systems, Holden-Day, 1964, 14.

FIGURE A1 - SOLUTIONS TO HILL'S EQUATION - IMPULSE EXCITATION OF STRENGTH 0.1 APPLIED TO A SYSTEM OF UNIT CIRCULAR FREQUENCY. INITIAL CONDITIONS INDICATED IN THE FOLLOWING CAPTIONS (A1(a), A1(b), A1(c)) ARE PREDICATED ON AN ASSUMED INITIAL ENERGY PER UNIT MASS OF 1/2.

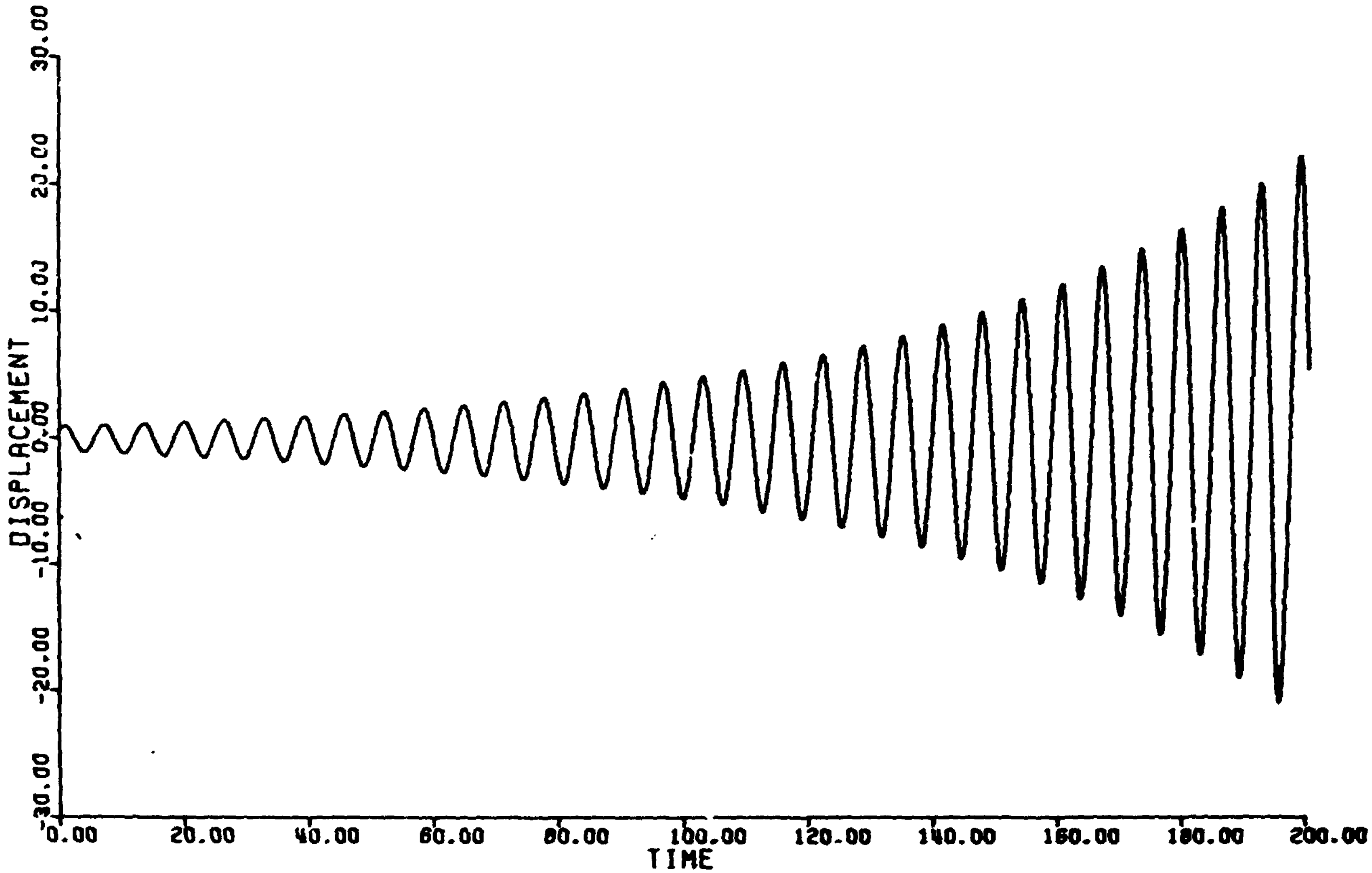


Figure A1(a) - Pure Growth: $y = c_1\phi_1 e^{\gamma t}$

Initial Conditions: $y(0) = 0.6892237$; $\dot{y}(0) = 0.7245468$

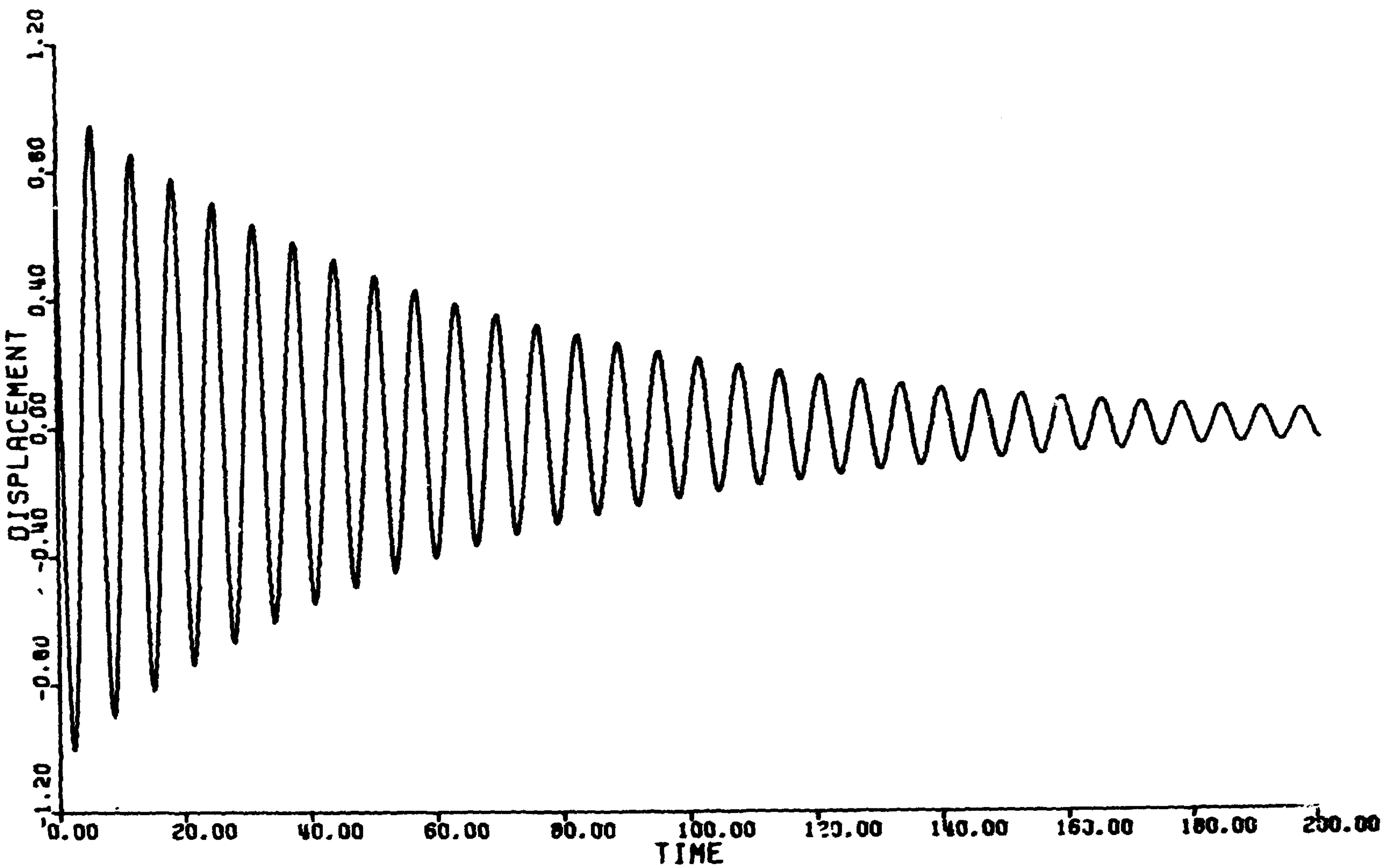


Figure A1(b) - Pure Decay: $y = c_2\phi_2 e^{-\gamma t}$
Initial Conditions: $y(0) = 0.7245462$; $\dot{y}(0) = -.6892250$

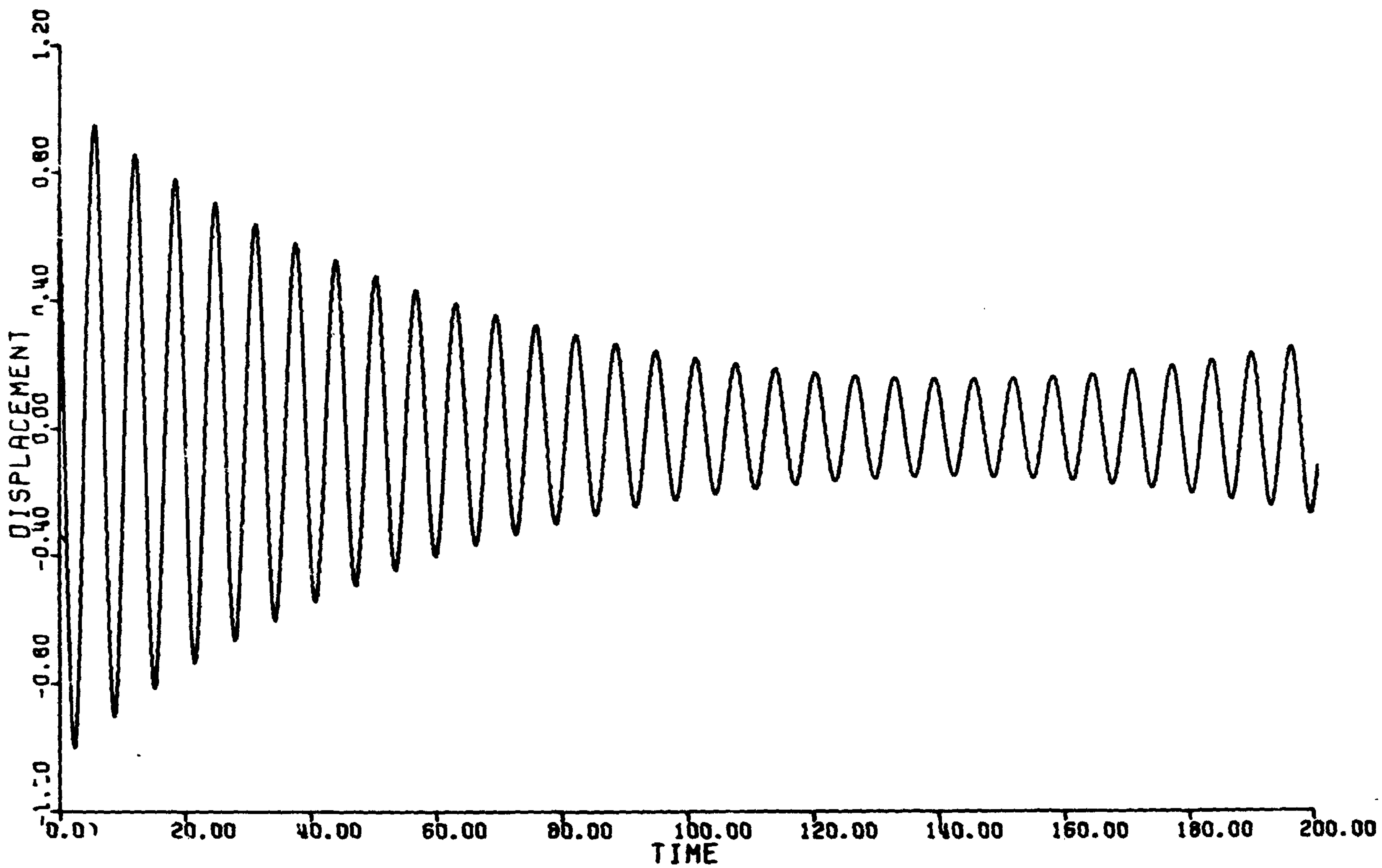


Figure A1(c) - General Case: $y = a\phi_1 e^{\gamma t} + b\phi_2 e^{-\gamma t}$
Initial Conditions: $y(0) = 0.7239993$;
$\dot{y}(0) = -.6889993$

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